Introduction to QFT

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Yesterday

- Field theories is like mechanics with few time dimensions
- To leading order fields are like many coupled harmonic oscillators
- We care about the non-harminic terms and for that we need perturbation theory

Perturbation theory

PT for 2 SHOs

$$V(x, y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- We assume that α is small
- Recall the Fermi golden rule

$$P \propto |\mathcal{A}|^2 \times \mathrm{P.S.} \qquad \mathcal{A} \sim \langle f|\alpha x^2 y|i\rangle$$

- The relevant thing to calculate is the transition amplitude, $\mathcal{A}$.

1st and 2nd order PT

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In first order we care only about the states with the same energy

$$\mathcal{A}(i \to f) \sim \langle f|H_1|i\rangle \qquad E_f = E_i$$

- 2nd order pertubation theory probe the whole spectrum

$$\mathcal{A}(i \to f) \sim \sum_n \frac{\langle f|H_1|n\rangle\langle n|H_1|i\rangle}{E_n - E_f} \qquad E_f = E_i \qquad E_n \neq E_i$$

Transitions

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- Recall

$$x \sim a_x + a_x^\dagger \qquad y \sim a_y + a_y^\dagger$$

- For a given i, for what f we have $\mathcal{A} \neq 0$?

$$\mathcal{A} \sim \langle f|\alpha x^2 y|i\rangle$$

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- Since $H_1 \sim x^2 y$ we see that $\Delta n_y = \pm 1$ and $\Delta n_x = 0, \pm 2$
- What could you say if the perturbation was $x^2 y^3$?

Two SHOs with small α

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y \qquad \omega_y = 2\omega_x$$

- Consider $|i\rangle = |0,1\rangle$
- Since $\omega_y = 2\omega_x$ only $f = |2,0\rangle$ is allowed by energy conservation and by the perturbation

$$\mathcal{A} \sim \langle 2,0|\alpha x^2 y|0,1\rangle \sim \alpha\langle 2,0|(a_x + a_x^\dagger)(a_x + a_x^\dagger)(a_y + a_y^\dagger)|0,1\rangle$$

- a_y in y annihilates the y "particle" and $(a_x^\dagger)^2$ in x^2 creates two x "particles"
- It is a decay of a particle y into two x particles with width $\Gamma \propto \alpha^2$ and $\tau = 1/\Gamma$

Even More PT

$$H_1 = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

- Calculate $y \to 3x$ using 2nd order PT

$$\mathcal{A} \sim \langle 3,0,0|\mathcal{O}|0,1,0\rangle \qquad \mathcal{O} \sim \sum \frac{\langle 3,0,0|H_1|n\rangle\langle n|H_1|0,1,0\rangle}{E_n - E_{0,1,0}}$$

- Which intermediate states? $|1,0,1\rangle$ and $|2,1,1\rangle$
  - $\mathcal{A}_1 = |0,1,0\rangle \xrightarrow{\beta} |1,0,1\rangle \xrightarrow{\alpha} |3,0,0\rangle$
  - $\mathcal{A}_2 = |0,1,0\rangle \xrightarrow{\alpha} |2,1,1\rangle \xrightarrow{\beta} |3,0,0\rangle$
- The total amplitude is then

$$\mathcal{A} \propto \alpha\beta\left(\frac{\#}{\Delta E_1} + \frac{\#}{\Delta E_2}\right)$$

Closer look

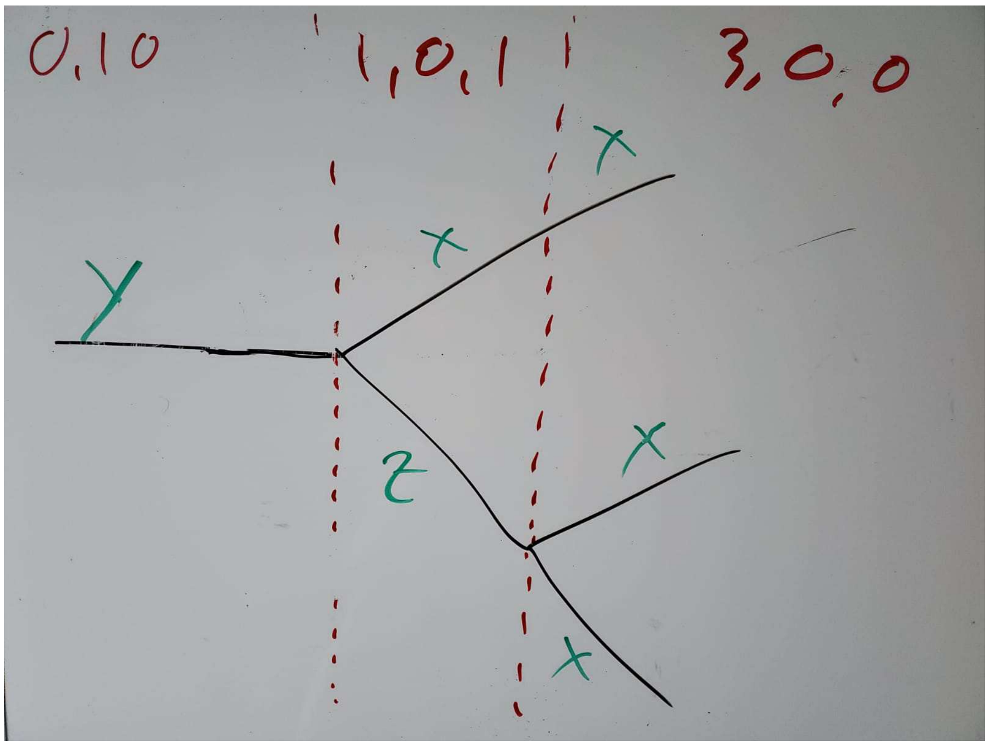
$$V' = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

- $\mathcal{A}_1 = |0,1,0\rangle \xrightarrow{\beta} |1,0,1\rangle \xrightarrow{\alpha} |3,0,0\rangle$
- $\mathcal{A}_2 = |0,1,0\rangle \xrightarrow{\alpha} |2,1,1\rangle \xrightarrow{\beta} |3,0,0\rangle$

First term

$$V' = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

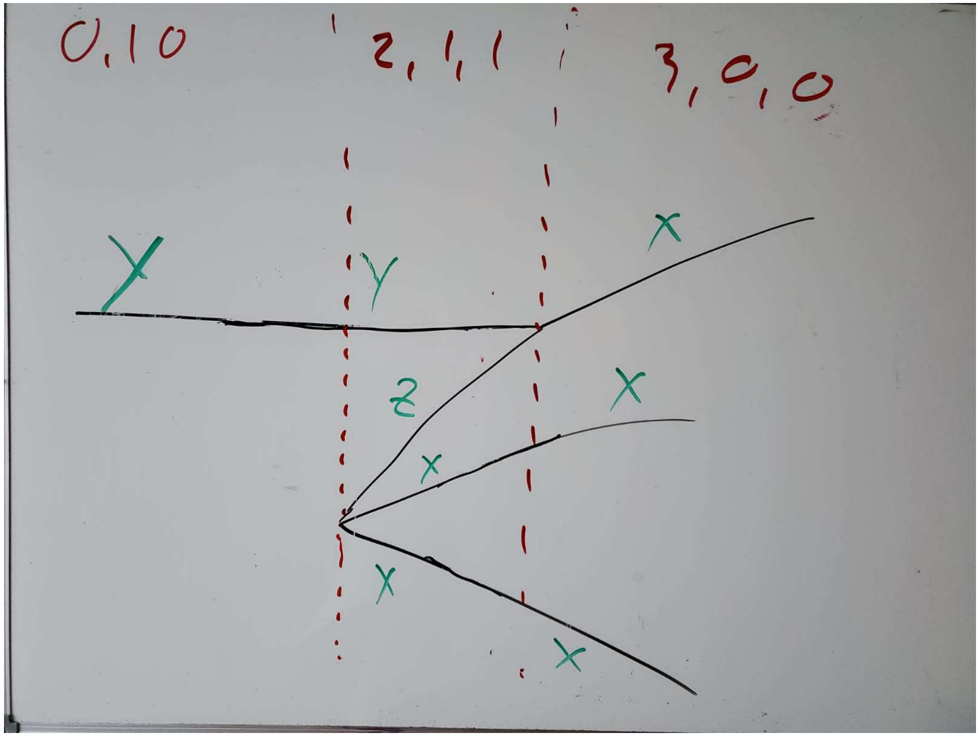
- $\mathcal{A}_1 = |0,1,0\rangle \xrightarrow{\beta} |1,0,1\rangle \xrightarrow{\alpha} |3,0,0\rangle$



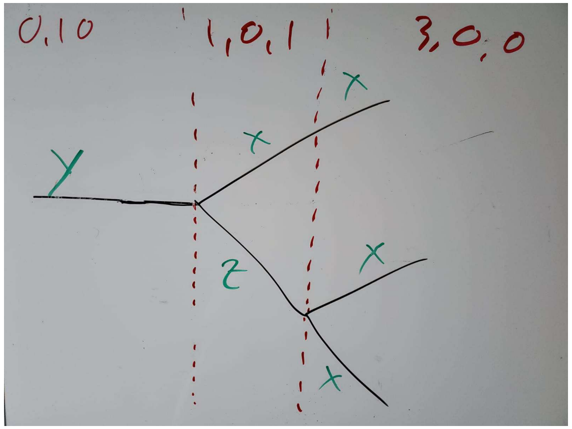
Second term

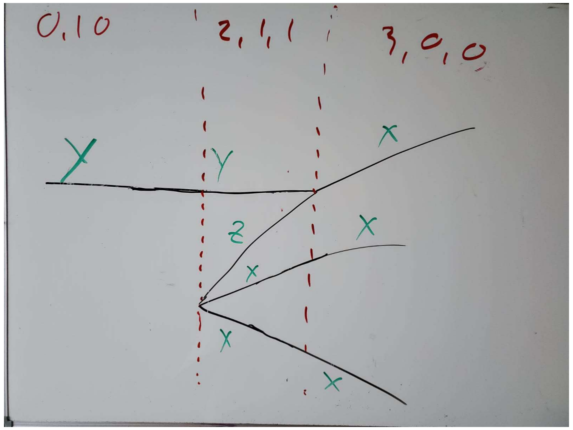
$$V' = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

- $\mathcal{A}_2 = |0,1,0\rangle \xrightarrow{\alpha} |2,1,1\rangle \xrightarrow{\beta} |3,0,0\rangle$



Sum them up





(HW) Show that the sum of the two diagrams give

$$\frac{1}{\omega_z^2 - q^2}$$

where q is the “energy flow” in the “z line”

- In our case $q = 2$ and $\omega_z = 10 \;\Rightarrow\; 10^2 - 2^2 = 12 \times 8$

General method for SHO PT

- Every x^n term with $n \geq 3$ is a vertex
- We write all the ways to get from "in" to "out"
- Each amplitude is the product of the couplings and the "off-shell" intermediate states

$$\frac{1}{\omega^2 - q^2}$$

- There are few more rules
- We add all amplitude square them and use the Fermi Golden rule

Feynman diagrams

Using PT for fields

- For one SHO we have $x \sim a + a^\dagger$
- For many SHOs we have $x_i \sim a_i + a_i^\dagger$
- For fields we then have $\phi \sim \int \left[a(k) + a^\dagger(k)\right] \, dk$

Perturbation theory for fields is a generalization of that of SHO

- $\omega \to p_\mu$
- $\omega^2 \to m^2$
- We can have any energy (but one mass)

Calculations

- We usually care about $1 \to n$ or $2 \to n$ processes
- We need to make sure we have energy conservation
- External (Internal) particles are called on(off)–shell
  - On-shell: $E^2 = p^2 + m^2$
  - Off-shell: $E^2 \neq p^2 + m^2$
- $\mathcal{A} =$ the product of all the vertices and internal lines
- Each internal line with q^μ gives suppression

$$\frac{1}{m^2 - q^2}$$

- There are many more rules to get all the factors right

Examples of amplitudes

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\Gamma(Z \to XY)$$

- Draw the diagram and estimate the amplitude

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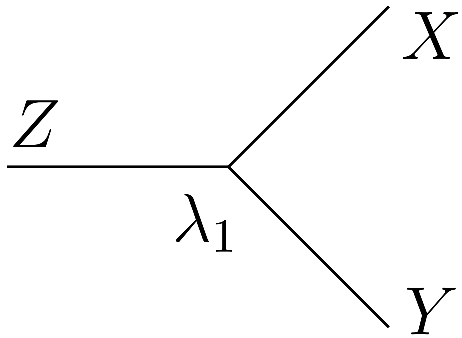

$$\mathcal{A} \propto \lambda_1$$

Examples of amplitudes (2)

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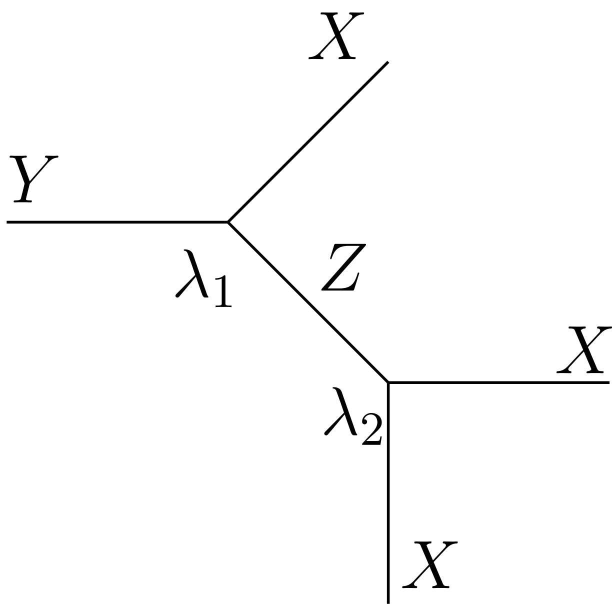

$$\mathcal{A} \propto \lambda_1 \lambda_2 \times \frac{1}{\Delta E_Z^2}$$

$$= \lambda_1 \lambda_2 \times \frac{1}{m_Z^2 - q^2}$$

Examples of amplitudes (HW)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\sigma(XX \to XY)$$

- Draw the diagram and estimate the amplitude

Some summary

- Quadratic terms describe free fields. Free particles cannot be created nor decay
- We use perturbation theory were terms with 3 or more fields in $\mathcal{L}$ are considered small
- These terms can generate and destroy particles and give dynamics
- Feynman diagrams are a tool to calculate transition amplitudes. They are a cool way to do PT for harmonic oscillators
- Many more details are needed to get calculation done
- Once calculations and experiments to check them are done, we can test our theory

Symmetries

How to “built” Lagrangians

- $\mathcal{L}$ is:
  - The most general one that is invariant under some symmetries
  - We work up to some order (usually 4)
- We need the following input:
  - What are the symmtires we impose
  - What DOFs we have and how they transform under the symmtry
- The output is
  - A Lagrangian with N parameters
  - We need to measure its parameters and test it

Symmetries and representations

Example: one particle in 3d real space in classical mechanics

- We require that $\mathcal{L}$ is invariant under rotation
- We assign the 3 DOF into a vector, $r = (x, y, z)$.
- We construct invariants from these DOFs. They are called singlets or scalars

$$C \equiv r \cdot r$$

- We then require that V is a function of C only

Generalizations

- In mechanics, $\vec{r}$ lives in 3d real space and is a vector
- Fields do not live in real space. They live in some mathematical space
- They also do not have to be vectors, but can be scalars or tensors
- The idea is similar to what we did in mechanics
  - We require $\mathcal{L}$ to be invariant under rotation in that mathematical space
  - Thus $\mathcal{L}$ depends only on combinations of fields that form singlets
- All this is related to a subject called Lie groups
- We usually care about $SO(N)$, $SU(N)$ and $U(1)$

Combining representations

- It is all about generating singlets
- We all know that we can combine vectors in real space to generate singlets
- We also know how to make a spin zero from 2 spin half spinors (spin zero is a singlet!)
- There is a generalization of this procedure to any mathematical space
- As of now, all we need to know are $SU(3)$, $SU(2)$ and $U(1)$

Invariant of complex numbers

- $U(1)$ is rotation in 1d complex space
- Each complex number, X comes with a real number q_X that tells us how much it rotates
- When we rotate the space by an angle θ, the number rotates as
$$X \to e^{iq_X\theta}X$$
- Consider $q_X = 1,\ q_Y = 2,\ q_Z = 3$ and write 3rd and 4th order invariants

$$XX^*YY^* \qquad X^2Y^*$$

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$$XX^*YY^* \qquad X^2Y^* \qquad XYZ^* \qquad X^3Z^* \qquad Y^2X^*Z^*$$

$SU(2)$

- $U(2)$ is rotation in 2d complex space. We have $U(2) = SU(2) \times U(1)$
- $SU(2)$ is localy the same as rotation in 3d real space
- Rotations in this space are non-Abelian (non-commutative)
- It depends on the representation: scalar, spinor, vector
- Spin in QM is described by SU(2) rotations, so we use the same language to describe it
- For the SM all we care is that $1/2 \times 1/2 \ni 0$ so we know how to generate singlets
- How can we generate invarints from spin $1/2$ and spin $3/2$?

$SU(3)$

- $U(3)$ is rotation in 3d complex space. We have $U(3) = SU(3) \times U(1)$
- The representations we care about are singlets, triplets and octets
- Unlike $SU(2)$, in $SU(3)$ we have complex representations, 3 and $\bar{3}$
- The three quarks form a triplet (the three colors)
- To form a singlet we need to know that

$$3 \times \bar{3} \ni 1 \qquad 3 \times 3 \times 3 \ni 1$$

- This is why we have baryons and mesons

A game

A game calls "building invariants"

- Symmetry is $SU(3) \times SU(2) \times U(1)$
  - $U(1)$: Add the numbers ($\bar{X}$ has charge $-q$)
  - $SU(2)$: $2 \times 2 \ni 1$ and recall that 1 is a singlet
  - $SU(3)$: we need $3 \times \bar{3} \ni 1$ and $3 \times 3 \times 3 \ni 1$
- Fields are

$$Q(3,2)_1 \qquad U(3,1)_4 \qquad D(3,1)_{-2} \qquad H(1,2)_3$$

- What 3rd and 4th order invariants can we built?

$$(HH^*)^2 \qquad H^3 \qquad UDD \qquad QUD \qquad HQU^*$$

- HW: Find more invariants

More on symmetires

Lorentz invariants

- The representations we care about are
  - Singlet: Spin zero (scalars, denote by ϕ)
  - LH and RH fields: Spin half (fermions, $\psi_L,\ \psi_R$)
  - Vector: Spin one (gauge boson, denote by A_μ)
- Fermions are more complicated

$$\mathcal{L} \sim \bar{\psi}\partial_\mu \gamma^\mu \psi$$

- Since $\mathcal{L}$ has dimension 4, ψ is dimension $3/2$
- For fermions when we expand up to 4th order we can have at most two fermion fields
- Under Lorentz, the basic fields are left-handed and right-handed. A mass term must involve both $m\bar{\psi}_L\psi_R$

Local symmetry

Basic idea: rotations depend on x and t

$$\phi(x_\mu) \to e^{iq\theta}\phi(x_\mu) \quad \xrightarrow{local} \quad \phi(x_\mu) \to e^{iq\theta(x_\mu)}\phi(x_\mu)$$

- It is kind of logical and we think that all imposed symmetries in Nature are local
- The kinetic term $|\partial_\mu \phi|^2$ in not invariant
- We want a kinetic term (why?)
- We can save the kinetic term if we add a field that is
  - Massless
  - Spin 1
  - Adjoint representation: $q = 0$ for $U(1)$, triplet for $SU(2)$, and octet for $SU(3)$

Gauge symmetry

- Fermions are called matter fields. What they are and their representation is an input
- Gauge fields are known as force fields
- Their couplings is via thekinetic termsand it is set by the charge/represenation ofthe field

Local symmetries $\Rightarrow$ force fields

Accidental symmetries

- We only impose local symmetries
- Yet, because we truncate the expansion, we can get symmetries as output
- What is the symmetry that makes A independet of ω for the SHO?
- They are global, and are called accidental
- Example: $U(1)$ with $X(q=1)$ and $Y(q=-4)$

$$V(XX^*, YY^*) \Rightarrow U(1)_X \times U(1)_Y$$

  - X^4Y breaks this symmetry
- In the SM baryon and lepton numbers are accidental symmetries

SSB

Breaking a symmetry

SSB

- By choosing a ground state we break the symmetry
- We choose to expend around a point that does not respect the symmetry
- PT only works when we expand around a minimum

What is the different between a broken symmetry and no symmetry?

SSB implies relations between parameters

SSB

Symmetry is $x \to -x$ and we keep up to x^4

$$f(x) = a^2x^4 - 2b^2x^2 \qquad x_{\min} = \pm b/a$$

We choose to expand around $+b/a$ and use $u \to x - b/a$

$$f(x) = 4b^2u^2 + 4bau^3 + a^2u^4$$

- No $u \to -u$ symmetry
- The $x \to -x$ symmetry is hidden
- A general function has 3 parameters $c_2u^2 + c_3u^3 + c_4u^4$
- SSB implies a relation between them

$$c_3^2 = 4c_2c_4$$

SSB in QFT

- When we expand the field around a minimum that is not invariant under a symmetry

$$\phi \to v + h$$

- It breaks the symmetries that ϕ is not a singlet under
- Masses to other fields via Yukawa interactions

$$\phi X^2 \to (v+h)X^2 = vX^2 + \ldots$$

Building Lagrangians

- Choosing the generalized coordinates (fields)
- Imposing symmetries and how fields transform (input)
- The Lagrangian is the most general one that obeys the symmetries
- We truncate it at some order, usually fourth

The SM

The SM

Input: Symmetries and fields

- Symmetry: 4d Poincare and

$$SU(3)_C \times SU(2)_L \times U(1)_Y$$

- Fields:
  - 3 copies of QUDLE fermions

$$Q_L(3,2)_{1/6} \quad U_R(3,1)_{2/3} \quad D_R(3,1)_{-1/3}$$
$$L_L(1,2)_{-1/2} \quad E_R(1,1)_{-1}$$

  - One scalar

$$\phi(1,2)_{+1/2}$$

Then Nature is described by