Lattice QCD results and tau measurements exchange: important inputs for precision tests

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Hadronic τ decays require a non-perturbative approach to QCD

this talk: predictions from Lattice Field theories

(disclaimer) selection of a few topics, far from complete

1. Lattice QCD
2. Rates from Lattice QCD
3. Hadronic τ decays: strange sector
4. Hadronic τ decays: light sector

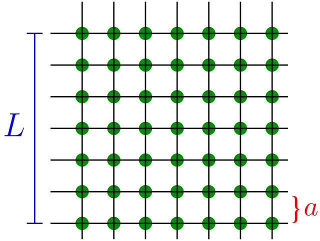
Lattice field theories

Mathematically sound non-perturbative formulation of QCD

lattice spacing a → regulate UV divergences

finite size L → infrared regulator

Continuum theory $a \to 0$, $L \to \infty$



Euclidean metric → Boltzman interpretation of path integral

$$\langle O \rangle = \mathcal{Z}^{-1} \int [DU] e^{-S[U]} O(U) \approx \frac{1}{N} \sum_{i=1}^{N} O[U_i]$$

Very high dimensional integral → Monte-Carlo methods

Lattice QCD

We start from QCD Lagrangian with N_f flavors: $\mathcal{L}(g_0, \{am_q\})$

dimensionless bare coupling g_0

N_f dimensionful quark masses $\{am_q\}$

Sacrifice $N_\mathrm{f} + 1$ input quantities makes LQCD predictive

typically hadron masses π^-, K^-, Ω^-

often pion/kaon decay constant instead of m_Ω

Primary objects in LQCD are Euclidean correlators

physical quantities obtained from their manipulation

typically energies + matrix elements of low-lying states

e.g. $m_\pi, m_p, \pi \to 0, K \to \pi$

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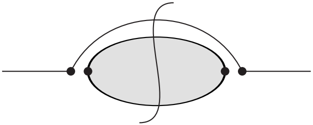
Phenomenology

1. Lattice QCD calculation of a quantity
   - statistical errors
   - (lattice) systematic errors
     - possible contaminations from excited states
     - discretization effects
     - finite volume, quark mass dependence

2. Lattice QCD $\neq$ Standard Model
   - (SM) systematic errors
     - QED effects, strong isospin breaking
     - effects of heavy quarks

3. Experimental precision

$$\mathcal{M}_f(P,q,p_1\cdots p_{n_f}) = \frac{G_F V_{ud}}{\sqrt{2}}\,\bar{u}_\nu(-q)\gamma^L_\mu u_\tau(P)\;\langle \text{out}, p_1\cdots p_{n_f}|\mathcal{J}^-_\mu(0)|0\rangle$$



$$d\Gamma = \frac{1}{4m}d\Phi_q \sum_f d\Phi_f \sum_{\text{spin}} |\mathcal{M}_f|^2$$

$$= \frac{1}{4m}d\Phi_q \frac{G_F^2|V_{ud}|^2}{2}\mathcal{L}_{\mu\nu}(P,q)\;\rho^{\mathrm{w}}_{\mu\nu}(p)$$

Transverse and longitudinal components $I = L, T$

Charged spectral densities isospin limit $= \rho_I^{\mathrm{w},0}$ $\qquad \left[d\Phi_q = \frac{d^3q}{(2\pi)^3 2\omega_q}\right]$

$$\frac{d\Gamma(s)}{ds} = G_F^2|V_{ud}|^2\frac{m^3}{16\pi}\sum_I \kappa_I(s)\,\theta(m_\tau^2 - s)\,\rho_I^{\mathrm{w},0}(s)$$

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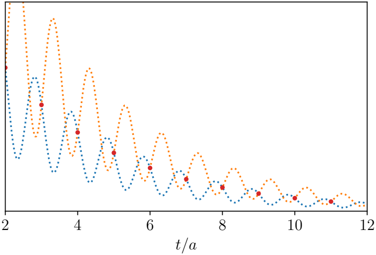
Time-like processes

Euclidean correlators

Rotation to Euclidean metric $\leftarrow$ Monte Carlo methods



finite noisy data $\rightarrow$ no analytic continuation back to Minkowski

so what physical information in Euclidean correlators?

Toy example:

1. $\widetilde{J}(t)$ scalar current w/ zero total momentum
2. Hamiltonian H, $H|n\rangle = E_n|n\rangle$
3. $\langle \widetilde{J}(t)\, \widetilde{J}(0)\rangle = \langle 0|\widetilde{J}(0)\, e^{-tH}\, \widetilde{J}(0)|0\rangle = \int d\omega\, e^{-t\omega}\, \rho(\omega)$

Spectral density contains physical information

experiment $\rightarrow$ spectral densities $\leftarrow$ Lattice correlators

Inverse Laplace

Method

Lattice correlator $\langle \widetilde{J}(t)\, \widetilde{J}(0) \rangle = \int d\omega\, e^{-\omega t}\, \rho(\omega)$

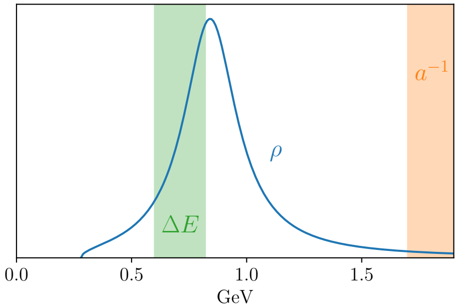
Inverse Laplace $[e^{-\omega t}] \to [\kappa(\omega)]$

Physical observable $\Gamma = \int d\omega\, \kappa(\omega)\, \rho(\omega)$

Inversion of Laplace transform is ill-conditioned problem
- errors of Lattice correlators amplified, tend to explode
- regularization scheme is required at intermediate stage
  - regulator acts as a smearing kernel

A new frontier for Lattice QCD [HLT][Bailas et al][MB et al][more ..]
- inclusive (=all channels) smeared spectral densities
  - ✓ high-precision
- exclusive, e.g. $1 \to 2$
  - ✓ formalism [MB, Hansen][Hansen, Bulava][Tantalo, Patella]
  - numerical tests

1. up, down physical masses ✓ ← algorithmic + technological advances
   strange quark ✓, sea charm effects if small typically controlled



2. lattice cutoff typically $\in [1.7, 4]$ GeV

3. energy resolution $\frac{2\pi}{L} \approx 200$ MeV

4. stat errs grow exponentially at long distances

What is better (on paper) for Lattice QCD?
smeared $\rho = \int d\omega\, \rho(\omega^2)\, \kappa(\omega)$ w/ broad κ
possibly low-pass filter
→ inclusive τ rates perfect candidate

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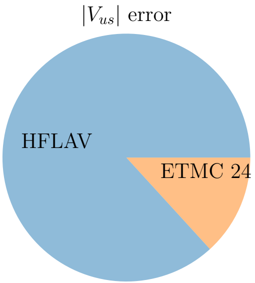
Hadronic τ decays

Recent first works on total rates [ETMC '23 '24] remarkable precision

1. Current $J_\mu = \bar{u}(V-A)_\mu s$
2. $\langle J_k(t,\vec{x})\, J_k^\dagger(0)\rangle = \int d\omega\, e^{-\omega t}\omega^2\, \rho_T(\omega^2$
3. $[e^{-\omega t}\omega^2] \to [\kappa_T]$
4. $\dfrac{R_{us}^{(\tau)}}{|V_{us}|^2} \propto \sum_{I=T,L} \int ds\, \kappa_I(s)\, \rho_I(s)$
5. experimental $R_{us}^{(\tau)} = \dfrac{\Gamma(\tau \to X_{us}\nu)}{\Gamma(\tau \to e\nu\bar{\nu})}$



Lattice QCD $< 1\%$ accuracy in isopin limit

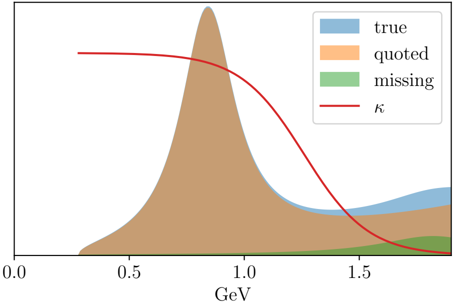
isospin-breaking missing

demonstrates potential of the method

A POSSIBLE SCENARIO

Gedanken experiment

Lattice spectral density (two-point correlator) fully inclusive
- comparison with fully inclusive experimental data
- known tensions in $|V_{us}|$ with exclusive modes $K_{\ell 3}$, $K_{\ell 2}$



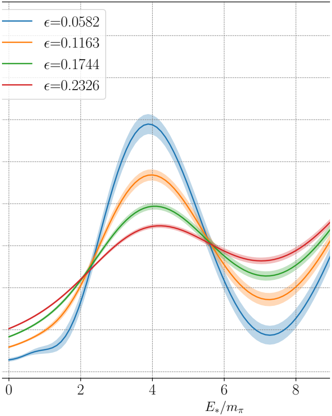
suppose systematics at high-energies

family of kernels κ w/ smooth cutoff
→ beneficial for Lattice QCD (finite-volume)
→ examine inclusivity problem

several kernels w/ similar goals already proposed [Boyle et al '10][Boito et al]

Example $N_\mathrm{f} = 2$

Example in toy model $N_\mathrm{f} = 2$ $m_\pi \approx 215$ MeV



isovector vector spectral density

smearing w/ Cauchy kernel $\dfrac{\epsilon}{(\omega - E_\star)^2 + \epsilon^2}$

ϵ in lattice units, $\epsilon \simeq 0.1 \approx 215$ MeV

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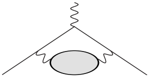
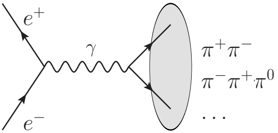
Hadronic input for $(g-2)_\mu$

Motivations



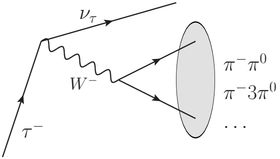
Hadronic Vacuum Polarization (HVP) contribution to a_μ

- largest error in theory prediction
- optical theorem relates it to $\sigma(e^+e^- \to \text{had})$
  - fragmented experimental situation in $ee \to \pi\pi$



EM current

Final states $I = 0, 1$ neutral



$V - A$ current

Final states $I = 1$ charged

[Alemani et al '98]
provided isospin-breaking corrections $\to$ τ relevant role in $(g-2)_\mu$

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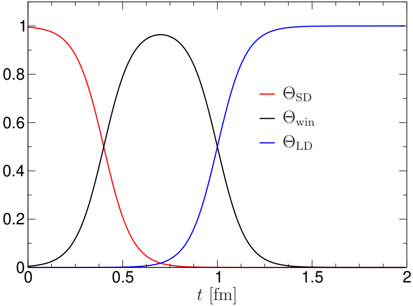
Euclidean τ windows

Euclidean time windows recently introduced in $(g-2)_\mu$ HVP roughly map onto energy windows

$$a_{\mu,\mathrm{win}} = \int dt\, \Theta_{\mathrm{win}}(t)\, w(t)\, C(t)$$



$a_{\mu,\mathrm{win}}$: 2π contribution from τ data

Example: ALEPH13
- error $< 1\%$ competitive w/ e^+e^-
- $\approx 40\%$ of error from $\frac{1}{\Gamma}\frac{d\Gamma}{ds}$
- $\approx 50\%$ of error from branch. ratio

advanced analysis [Davier et al]

Looking forward to new analysis from Belle II

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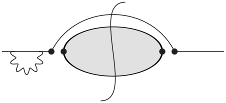
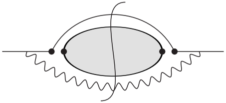
Isospin breaking effects

A possible strategy

analytic

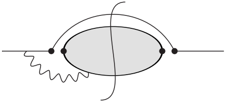


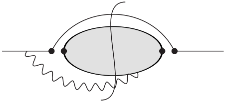


EFT for 2π

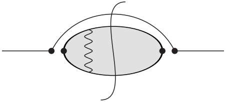
dispersive for 2π

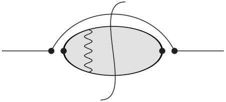
LQCD w/ inv-lap methods





LQCD+QED (inclusive)
in progress





Separation in these 3 classes is IR safe but gauge dependent

W REGULARIZATION

Short-distance effects

[Sirlin '82][Marciano, Sirlin '88][Braaten, Li '90]

Effective Hamiltonian $H_W \propto G_{\mathrm{F}} O_{\mu\nu}$

G_{F} low-energy constant; 4-fermion operator $O_{\mu\nu}$

At $O(\alpha)$ new divergences in EFT $\rightarrow$ need regulator, Z factors

$$\frac{1}{k^2} = \frac{1}{k^2 - m_W^2} - \frac{m_W^2}{k^2(k^2 - m_W^2)}$$

[Sirlin '78]

1. universal UV divergences re-absorbed in G_{F}
2. process-specific corrections in S_{EW}, like a Z factor

Effective Hamiltonian at $O(\alpha)$: $H_W \propto G_{\mathrm{F}} S_{EW}^{1/2} O_{\mu\nu}$

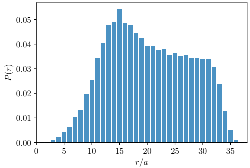
matching required as noted by [Carrasco et al '15][Di Carlo et al '19]

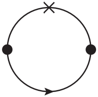
First results

Connected strong-isospin breaking

Ideas from stochastic locality [Lüscher '17][RBC/UKQCD '23][MB, Cé et al '23]

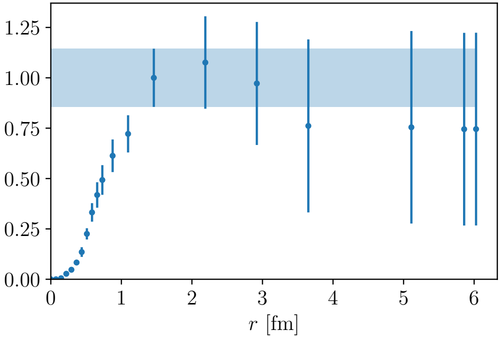
$O(10^3)$ point sources
$\rightarrow O(10^6)$ pairs





r = spatial separation vector and mass operators

t^4 interm. window [preliminary 96I]



Conclusions

Theory meets experiment

Lattice QCD in isospin limit very precise

- access to inclusive time-like smeared densities now possible
- isospin-breaking effects relevant and next target

Pheno impactful studies require manipulation of $d\Gamma/ds$, hence:

i. covariance matrices

ii. "details on photons" relevant
   - paired with correct isospin-breaking corrections from LQCD

iii. typically unit normalized rates $\frac{1}{\Gamma}\frac{d\Gamma}{ds}$ require branching fractions
   - improve determination of those?

Thanks for your attention

Numerical Inverse Laplace

Approximate solution $\sum_t g_t e^{-\omega t} = \kappa(\omega)$

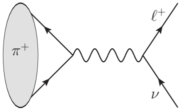
1. minimize norm $\int d\omega \big[\sum_t g_t e^{-\omega t} - \kappa(\omega) \big]^2$
2. define $A(t,t') = \int d\omega e^{-\omega(t+t')}$, $\quad f(t) = \int d\omega \kappa(\omega) e^{-\omega t}$
3. solution is $g_t = \sum_{t'} [A^{-1}]_{t,t'} f(t')$

A ill-conditioned $\to g_t$ useless in practice

Regulators:

1. covariance matrix [Backus, Gilbert '68][Hansen, Lupo, Tantalo '19]
2. Tikhonov [MB, Giusti, Saccardi '24]
   $W[\lambda] = A(1-\lambda) + \lambda B$ and evaluate $g_t = \sum_{t'} [W^{-1}]_{t,t'} f(t')$
3. gaussian processes as broader framework [Del Debbio et al '24]
4. truncation to fewer time-slices (improves cond. number of A)
   Chebyshev polynomials [Bailas, Hashimoto, Ishikawa '20]
   handful selection of points [Boito et al]

Decay constants



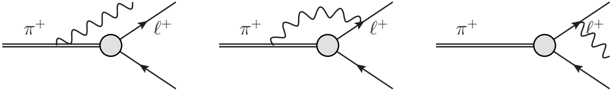
Leading-order in electro-weak (tree-level)

$$\Gamma^{(0)}(\pi^+ \to \ell^+\nu) = \frac{G_F^2}{8\pi}|V_{ud}|^2 f_\pi^2 m_\ell^2 m_\pi \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2$$

experimental rate very precise → NLO

Radiative corrections $\Gamma(\pi^+ \to \ell^+\nu[\gamma]) = \Gamma^{(0)}(\pi^+ \to \ell^+\nu)\left[1 + \delta_\pi\right]$

can be computed in ChPT



IR divergences properly cancel: universal short and long distance parts

structure-depedent parts more difficult in ChPT (large syst. errs)

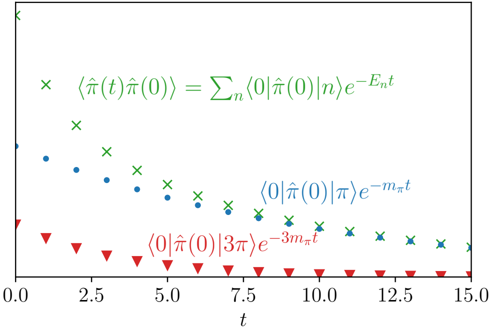
Masses, Matrix elements

Primary objects in LQCD are Euclidean correlators

e.g operator $\hat{\pi}$ w/ pion quantum numbers

Hamiltonian $\hat{H}$, $\hat{H}|n\rangle = E_n|n\rangle$

$$\langle\hat{\pi}(t)\,\hat{\pi}(0)\rangle = \langle 0|e^{\hat{H}t}\hat{\pi}(0)\,e^{-\hat{H}t}\,\hat{\pi}(0)\rangle = \sum_n |\langle 0|\hat{\pi}(0)|n\rangle e^{-E_n t}$$



$t \gg 0 \rightarrow$ energies + matrix elements of low-lying states

$\hat{H}$ = finite-volume Hamiltonian, care required e.g. w/ finite-volume $|3\pi\rangle$