Panel Dicussion: Precision tests of the Standard Model with Tau physics

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2024 Belle II Physics Week
«Tau and dark sector with Belle II»
October 14 - 17, 2024

Discussion topics

- What are the *new ideas* and *information* presented at this workshop?
- What should we focus on?
- Where can Tau physics play an important role?

Talks by *S. Banerjee*, *M. Bruno*, *M. Hoferichter*, *E.P.*, *S. Prell, P. Roig*

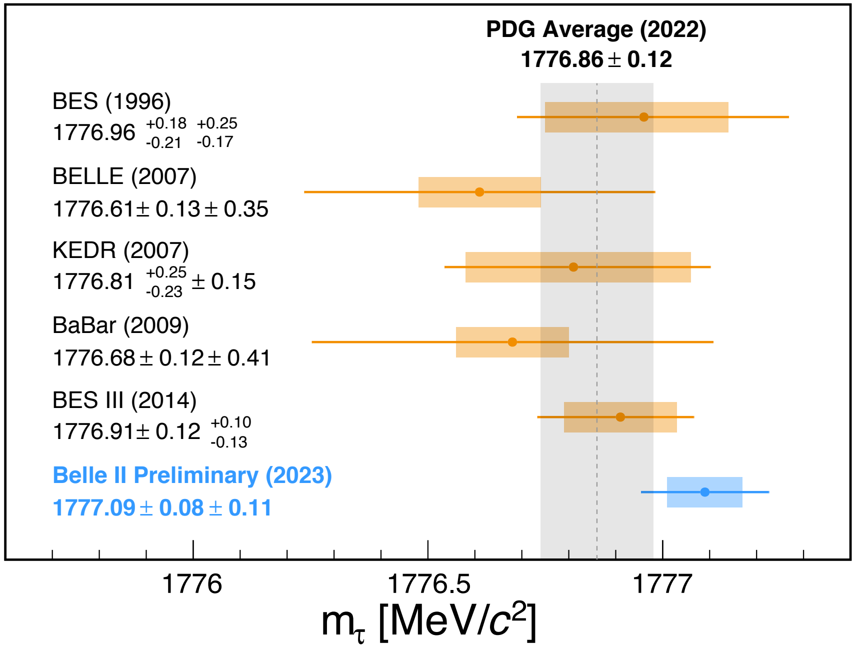
1. Leptonic τ decays

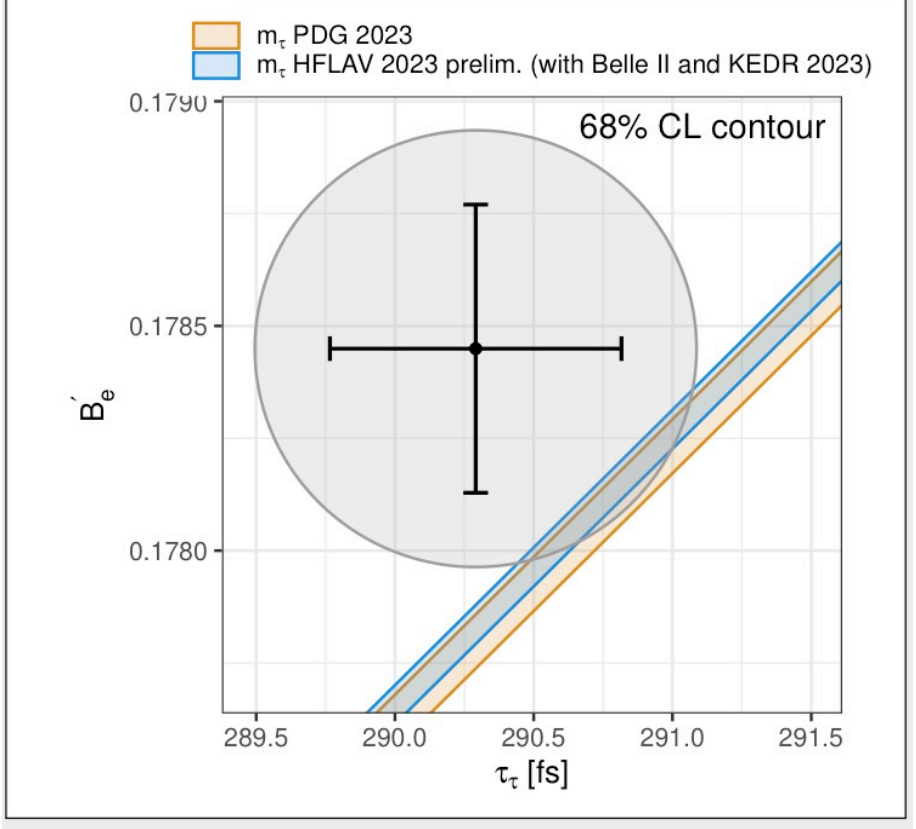
- Improve on m_τ measurement : fundamental parameter of the SM

⇨ Improve Lepton Universality test + $(g\text{-}2)_\tau$

$$B'(\tau \to e\bar{\nu}\nu) \approx B(\mu \to e\bar{\nu}\nu)\,\frac{m_\tau^5}{m_\mu^5}\,\frac{\tau_\tau}{\tau_\mu}$$

arXiv:2305.19116





- Measure the absolute Brs, they have not been updated since LEP

1. Leptonic τ decays

- For constraints on the *Lorentz structure*:

  ⇨ Michel parameters see talks by *S. Prell* and *P. Roig*

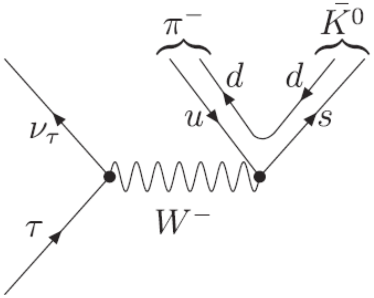
  One can constrain sterile neutrinos

- Prospects on $(g-2)_\tau$ with polarized beams ⇨ see *M. Hoferichter's* talk

2. Hadronic τ decays

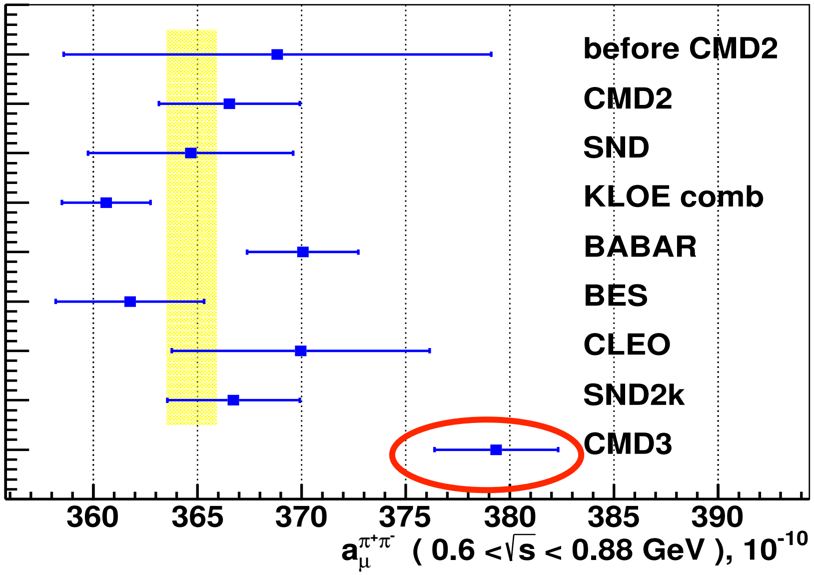
- Several anomalies where τ physics can help
  - Cabibbo angle anomaly: V_{us} extraction
  - CP asymmetry in $\tau \rightarrow K\pi\nu_\tau$

$$\boxed{A_Q = \frac{\Gamma\left(\tau^+ \rightarrow \pi^+ K_S^0 \bar{\nu}_\tau\right) - \Gamma\left(\tau^- \rightarrow \pi^- K_S^0 \nu_\tau\right)}{\Gamma\left(\tau^+ \rightarrow \pi^+ K_S^0 \bar{\nu}_\tau\right) + \Gamma\left(\tau^- \rightarrow \pi^- K_S^0 \nu_\tau\right)}}$$

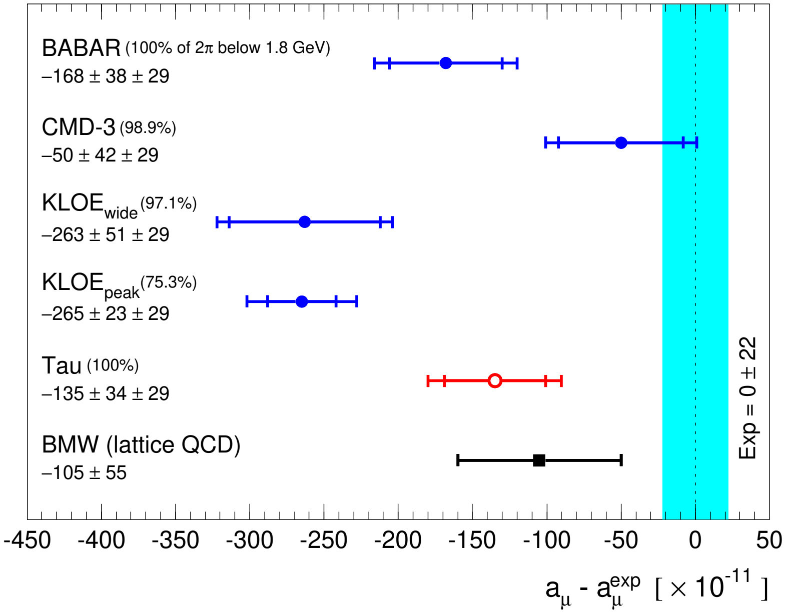


  - g-2 of the muon

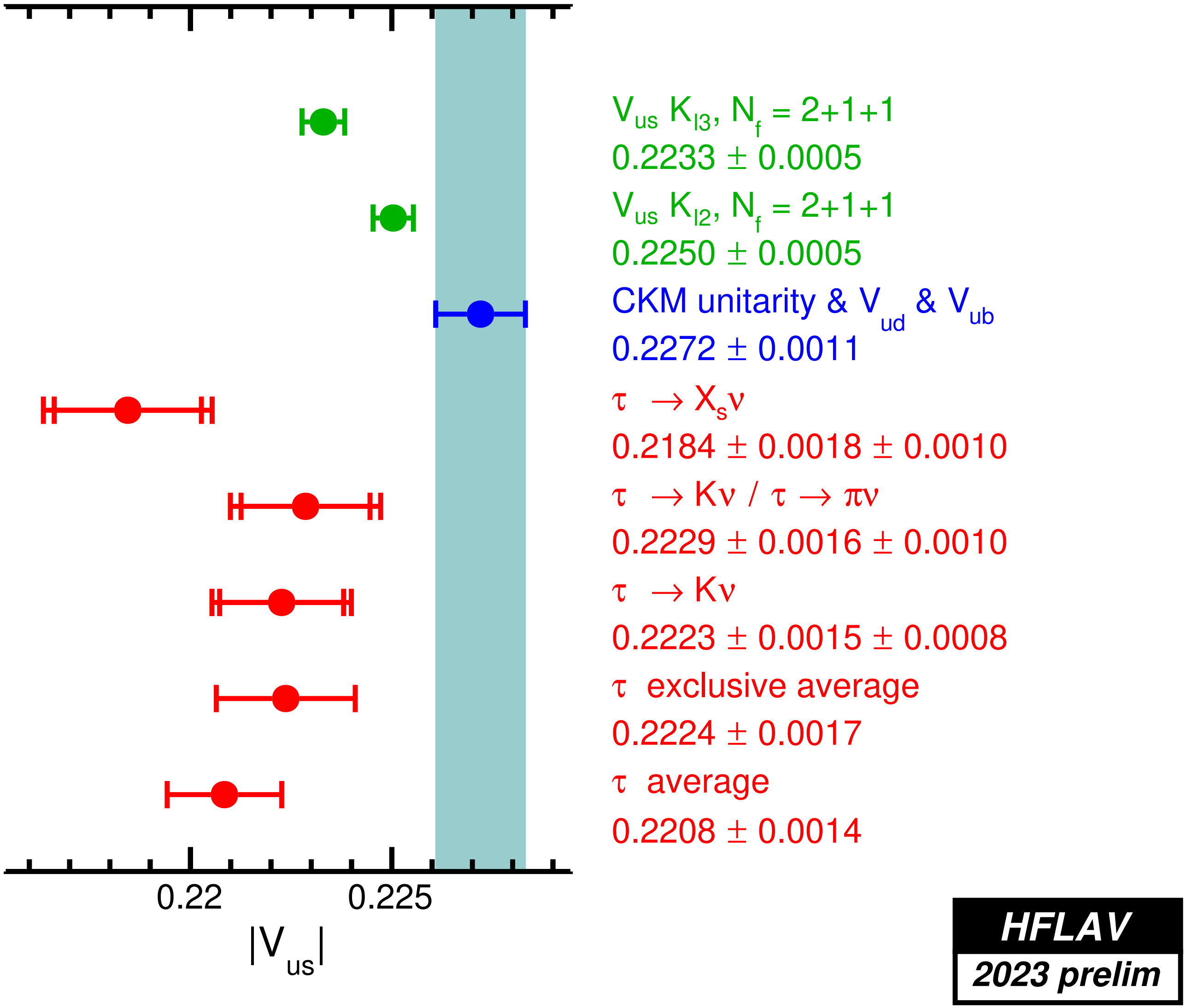
$\boxed{e^+e^- \rightarrow \pi^+\pi^-}$



Davier et al.'24



2.1 Important experimental inputs



2.1 Important experimental inputs

- Modes measured in the strange channel for $\tau \to s$:

HFLAV'23

Branching fraction	**HFLAV 2023 fit (%)**
$\mathcal{B}(\tau^- \to K^-\nu_\tau)$	0.6959 ± 0.0096
$\mathcal{B}(\tau^- \to K^-\pi^0\nu_\tau)$	0.4321 ± 0.0148
$\mathcal{B}(\tau^- \to K^-2\pi^0\nu_\tau\ (\mathrm{ex.}K^0))$	0.0634 ± 0.0219
$\mathcal{B}(\tau^- \to K^-3\pi^0\nu_\tau\ (\mathrm{ex.}K^0, \eta))$	0.0465 ± 0.0213
$\mathcal{B}(\tau^- \to \pi^-\overline{K}^0\nu_\tau)$	0.8375 ± 0.0139
$\mathcal{B}(\tau^- \to \pi^-\overline{K}^0\pi^0\nu_\tau)$	0.3810 ± 0.0129
$\mathcal{B}(\tau^- \to \pi^-\overline{K}^0 2\pi^0\nu_\tau\ (\mathrm{ex.}K^0))$	0.0234 ± 0.0231
$\mathcal{B}(\tau^- \to \overline{K}^0 h^- h^- h^+\nu_\tau)$	0.0222 ± 0.0202
$\mathcal{B}(\tau^- \to K^-\eta\nu_\tau)$	0.0155 ± 0.0008
$\mathcal{B}(\tau^- \to K^-\pi^0\eta\nu_\tau)$	0.0048 ± 0.0012
$\mathcal{B}(\tau^- \to \pi^-\overline{K}^0\eta\nu_\tau)$	0.0094 ± 0.0015
$\mathcal{B}(\tau^- \to K^-\omega\nu_\tau)$	0.0410 ± 0.0092
$\mathcal{B}(\tau^- \to K^-\phi(K^+K^-)\nu_\tau)$	0.0022 ± 0.0008
$\mathcal{B}(\tau^- \to K^-\phi(K^0_S K^0_L)\nu_\tau)$	0.0015 ± 0.0006
$\mathcal{B}(\tau^- \to K^-\pi^-\pi^+\nu_\tau\ (\mathrm{ex.}K^0, \omega))$	0.2924 ± 0.0068
$\mathcal{B}(\tau^- \to K^-\pi^-\pi^+\pi^0\nu_\tau\ (\mathrm{ex.}K^0, \omega, \eta))$	0.0388 ± 0.0142
$\mathcal{B}(\tau^- \to K^-2\pi^-2\pi^+\nu_\tau\ (\mathrm{ex.}K^0))$	0.0001 ± 0.0001
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~70% of the decay modes

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~70% of the decay modes

Up to ~90%
Including the
2π modes

⇨ Useful for V_{us} inclusive and exclusive

2.2 Lattice QCD

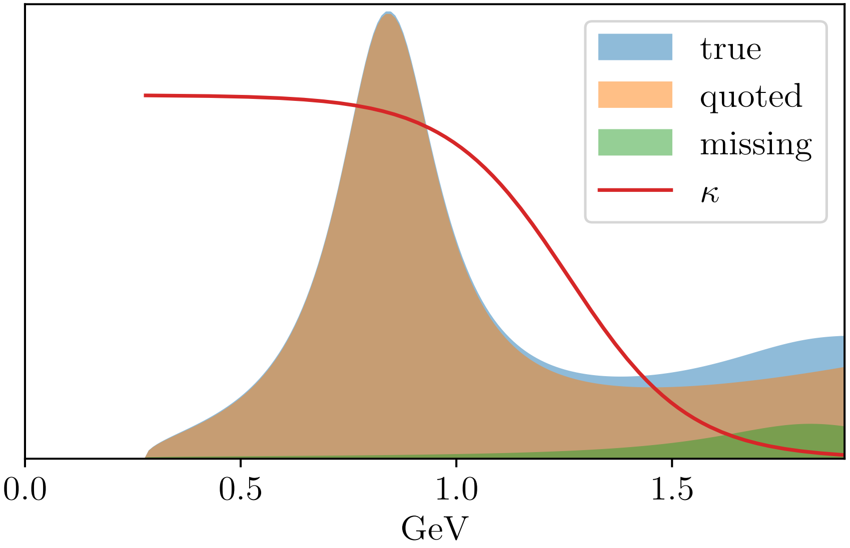
M. Bruno

ETMC'24

A POSSIBLE SCENARIO

Gedanken experiment

Lattice spectral density (two-point correlator) fully inclusive
comparison with fully inclusive experimental data
known tensions in $|V_{us}|$ with exclusive modes $K_{\ell 3}$, $K_{\ell 2}$



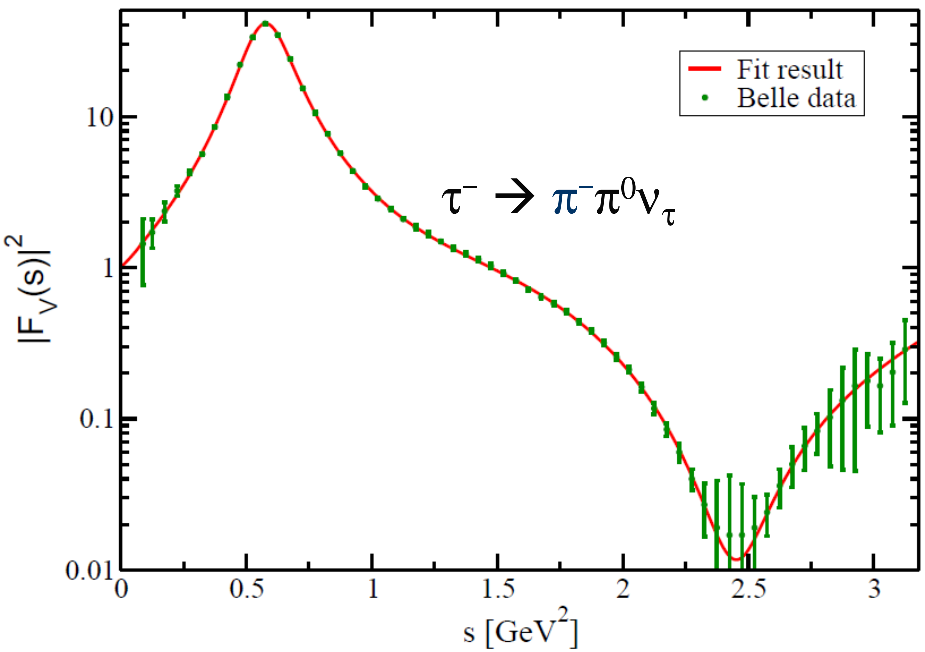
suppose systematics at high-energies

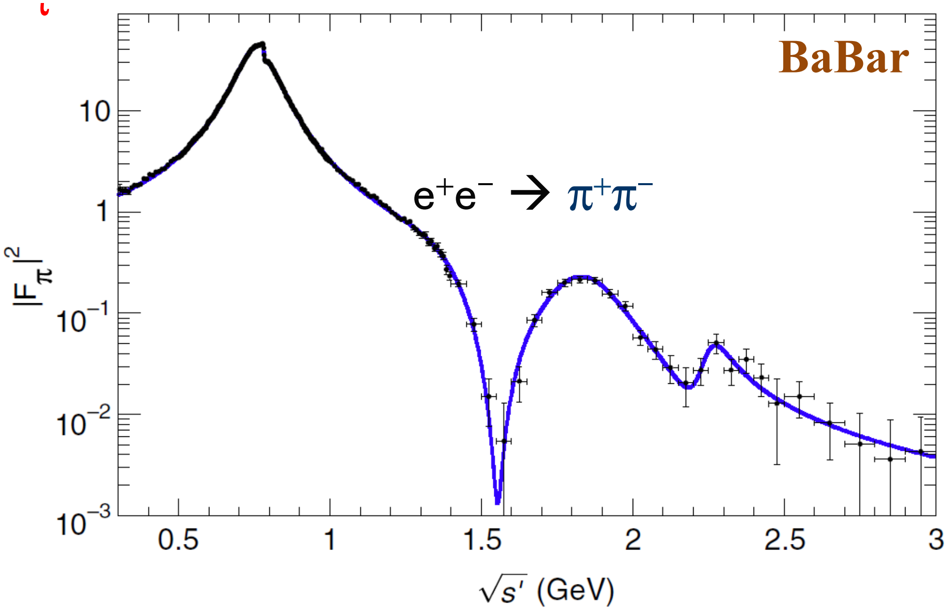
family of kernels κ w/ smooth cutoff
→ beneficial for Lattice QCD (finite-volume)
→ examine inclusivity problem

several kernels w/ similar goals already proposed [Boyle et al '10][Boito et al]

2.3 Exclusive hadronic Tau decays

- Key measurements:
  - $\pi\pi$ vector form factor for g-2 of the muon + also $e^+e^- \rightarrow \pi^+\pi^-$ with ISR





IB corrections should be precisely known (see talk by *M. Bruno*)

2.3 Exclusive hadronic Tau decays

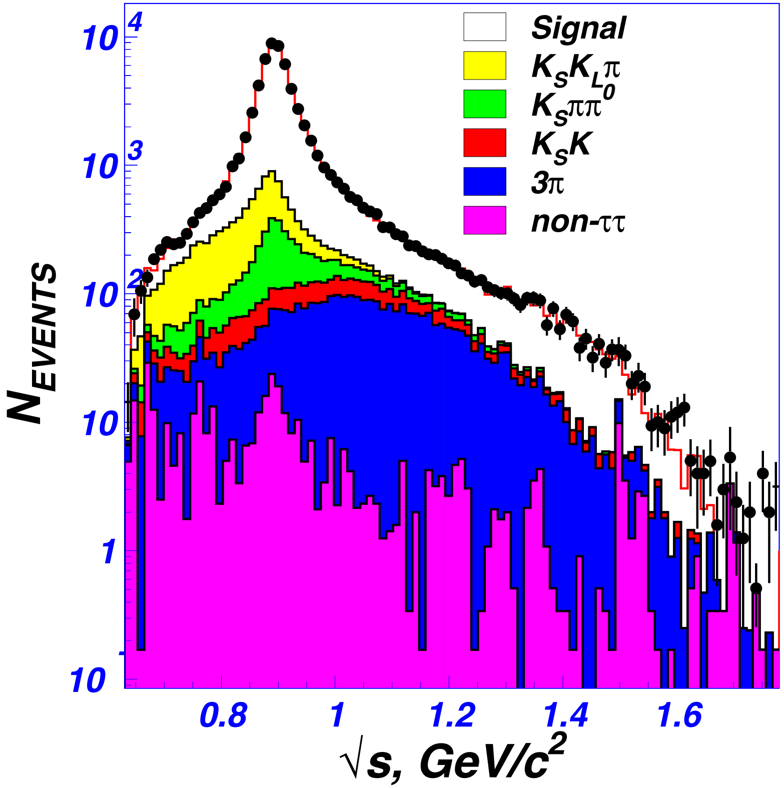
- Key measurements: Kπ invariant mass distribution + FB asymmetry ⇨ info on Kπ vector and scalar FFs: Crucial inputs for phenomenology

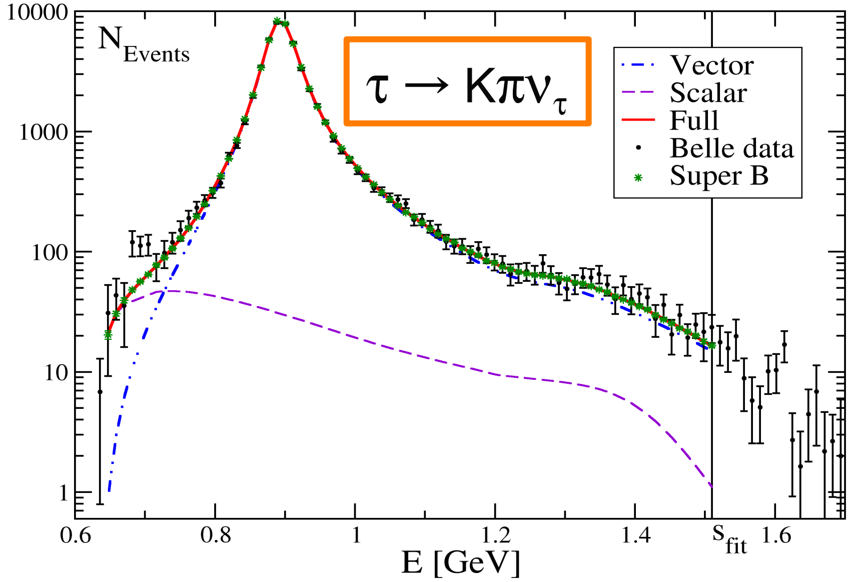
$$\langle \mathbf{K}\pi|\, \bar{\mathbf{s}}\gamma_\mu \mathbf{u}\, |0\rangle = \left[\left(p_K - p_\pi\right)_\mu - \frac{\Delta_{K\pi}}{s}\left(p_K + p_\pi\right)_\mu \right] f_+(s) + \frac{\Delta_{K\pi}}{s}\left(p_K + p_\pi\right)_\mu f_0(s)$$

vector

scalar

Belle'07





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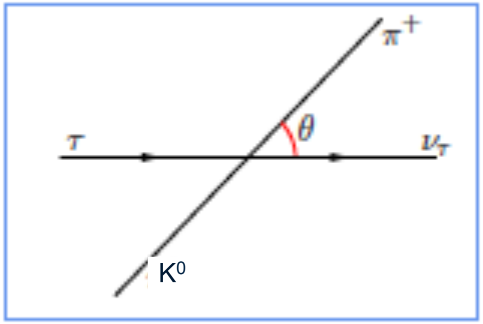
vector

scalar

$$\boxed{A_{\mathrm{FB}} = \frac{d\Gamma(\cos\theta) - d\Gamma(-\cos\theta)}{d\Gamma(\cos\theta) + d\Gamma(-\cos\theta)}}$$

Beldjoudi & Truong'94
Moussallam, B2TIP
Von Detten'21,
Rendon et al.'24



$$A_{FB}(s) = \frac{3\Delta_{\pi^+K^0}\sqrt{\lambda_{\pi^+K^0}(s)}\,|f_V^{K\pi}(s)||f_0^{K\pi}(s)|\cos\left(\delta_1^{1/2} - \delta_0^{1/2}\right)}{\underbrace{|f_V^{K\pi}(s)|^2\lambda_{\pi^+K^0}(s)}_{\text{vanishes at threshold}}(1 + 2s/m_\tau^2) + 3|f_0^{K\pi}(s)|^2\Delta^2_{\pi^+K^0}}.$$

→ *Never measured before!*

→ Kπ FFs: building block for many phenomenological analyses: B → *K** ll, B →Kπlν, D →Kππ, …

Theoretical improvements & Experimental needs

- Inclusion and calculations of Isospin breaking and EM effects which are crucial at the level of precision:

  ⇨ analytical (talk by *P. Roig*) and with lattice QCD (talk by *M. Bruno*)

  Measurement of $\tau \to PP\gamma\,\nu_\tau$ needed

  ⇨ test the *structure-dependent radiative* corrections

- Focus on Br with 1 K then Kπ then Kππ

- Invariant mass distribution

- Importance of providing efficiency corrected data with covariance matrix

- Collaboration between experimentalists and theorists is crucial

- Other ideas?

3. Back-up

$\tau \to K\pi\nu_\tau$ CP violating asymmetry: new physics

Devi, Dhargyal, Sinha'14
Cirigliano, Crivellin, Hoferichter'17

- We need a tensor interaction to get some interference:

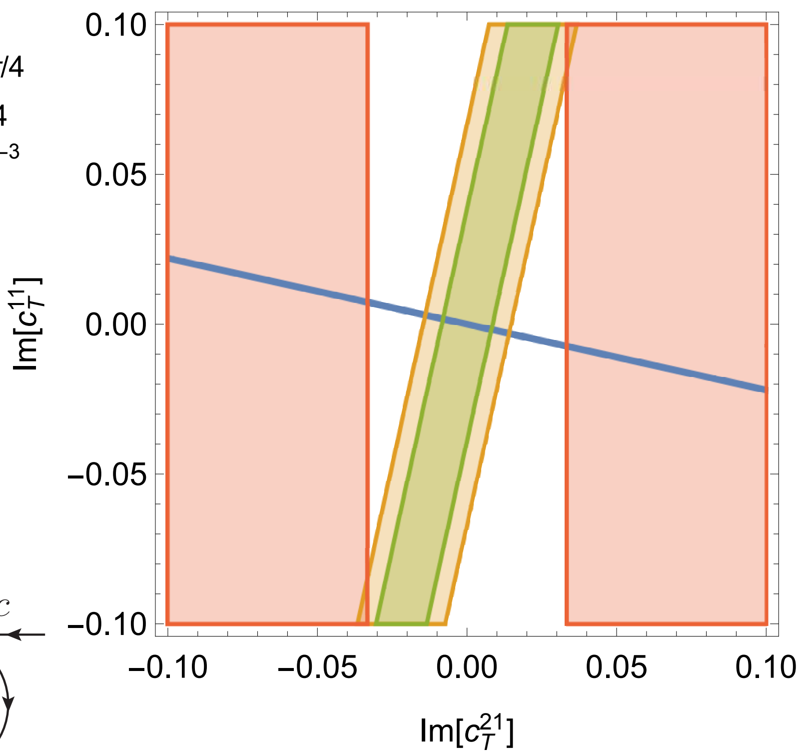
$$\boxed{\mathcal{H}_T^{eff} \equiv G'(\bar{s}\sigma_{\mu\nu}u)(\bar{\nu}_\tau(1+\gamma_5)\sigma^{\mu\nu}\tau)}$$

with $G' = \frac{G_F}{\sqrt{2}}C_T, \quad C_T = |C_T|e^{i\phi_T}$

- When integrating the interference term between vector and tenson does not vanish:

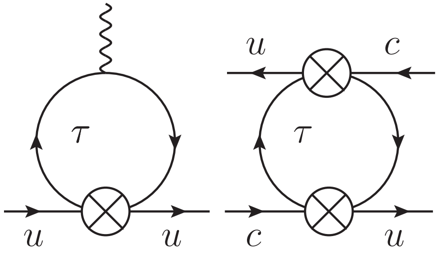
$$\frac{d\Gamma}{dQ^2} = \frac{d\Gamma_{SM}}{dQ^2} + \frac{d\Gamma_T}{dQ^2} + \frac{d\Gamma_{V-T}}{dQ^2}$$

$$\boxed{\frac{d\Gamma_{V-T}}{dQ^2} = G_F^2 \sin^2\theta_C \frac{m_\tau^3}{32\pi^3}\left(\frac{m_\tau^2 - Q^2}{m_\tau^2}\right)^2 \frac{q_1^3}{(Q^2)^{3/2}} \frac{Q^2}{m_\tau^2} \times |C_T||F_V(s)||F_T(s)|\cos\left(\delta_T(s) - \delta_V(s) + \phi_T\right)}$$



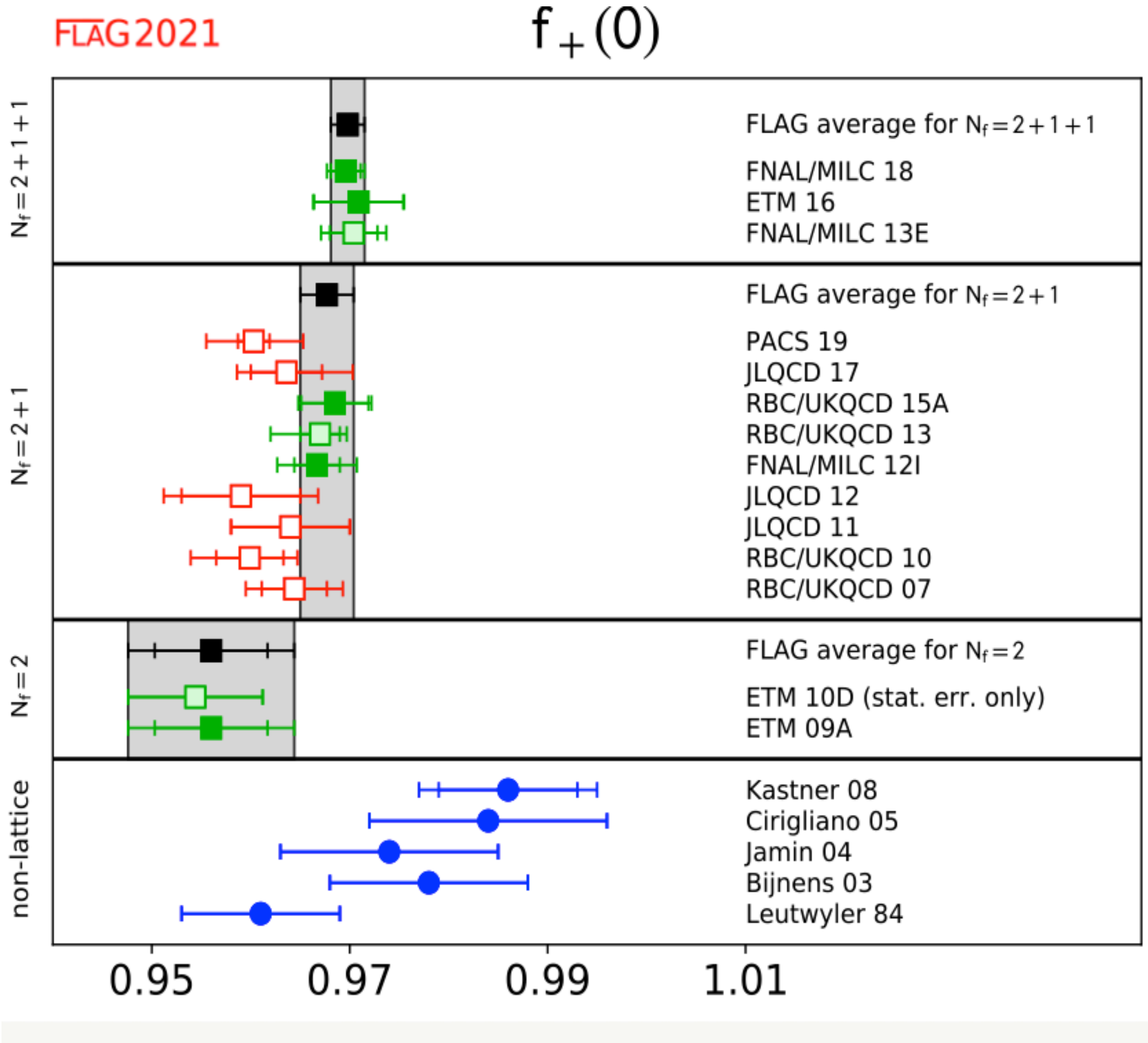
In conflict with bounds from neutron EDM and D$\bar{\text{D}}$ mixing

Cirigliano, Crivellin, Hoferichter'17



2.2 $f_+(0)$ from lattice QCD

- Recent progress on Lattice QCD for determining $f_+(0)$



$$f_+(0)^{FLAG21}_{N_f=2+1+1} = \mathbf{0.9698(17)}$$

0.18% uncertainty

to be compared to

$$f_+(0)^{FLAG16}_{N_f=2+1+1} = \mathbf{0.9704(32)}$$

$$f_+(0)^{2010}_{N_f=2+1} = \mathbf{0.959(50)}$$

Uncertainty divided by ~2 w/ 2016 and by 25 w/ 2011!

⇨ Lattice uncertainties at the same level as exp.

-3.2σ away from unitarity!

2011: $V_{us} = 0.2254(5)_{exp}(11)_{lat}$ → $V_{us} = 0.2231(4)_{exp}(4)_{lat}$

V_{us}/V_{ud} from $K_{\ell 2}/\pi_{\ell 2}$

$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi} = \left(\frac{\Gamma_{K_{\mu 2(\gamma)}} m_{\pi^\pm}}{\Gamma_{\pi_{\mu 2(\gamma)}} m_{K^\pm}}\right)^{1/2} \frac{1-m_\mu^2/m_{\pi^\pm}^2}{1-m_\mu^2/m_{K^\pm}^2}\left(1-\frac{1}{2}\delta_{\mathrm{EM}}-\frac{1}{2}\delta_{SU(2)}\right)$$

- Recent progress on radiative corrections computed on lattice:

Di Carlo et al.'19

- Main input hadronic input: f_K/f_π

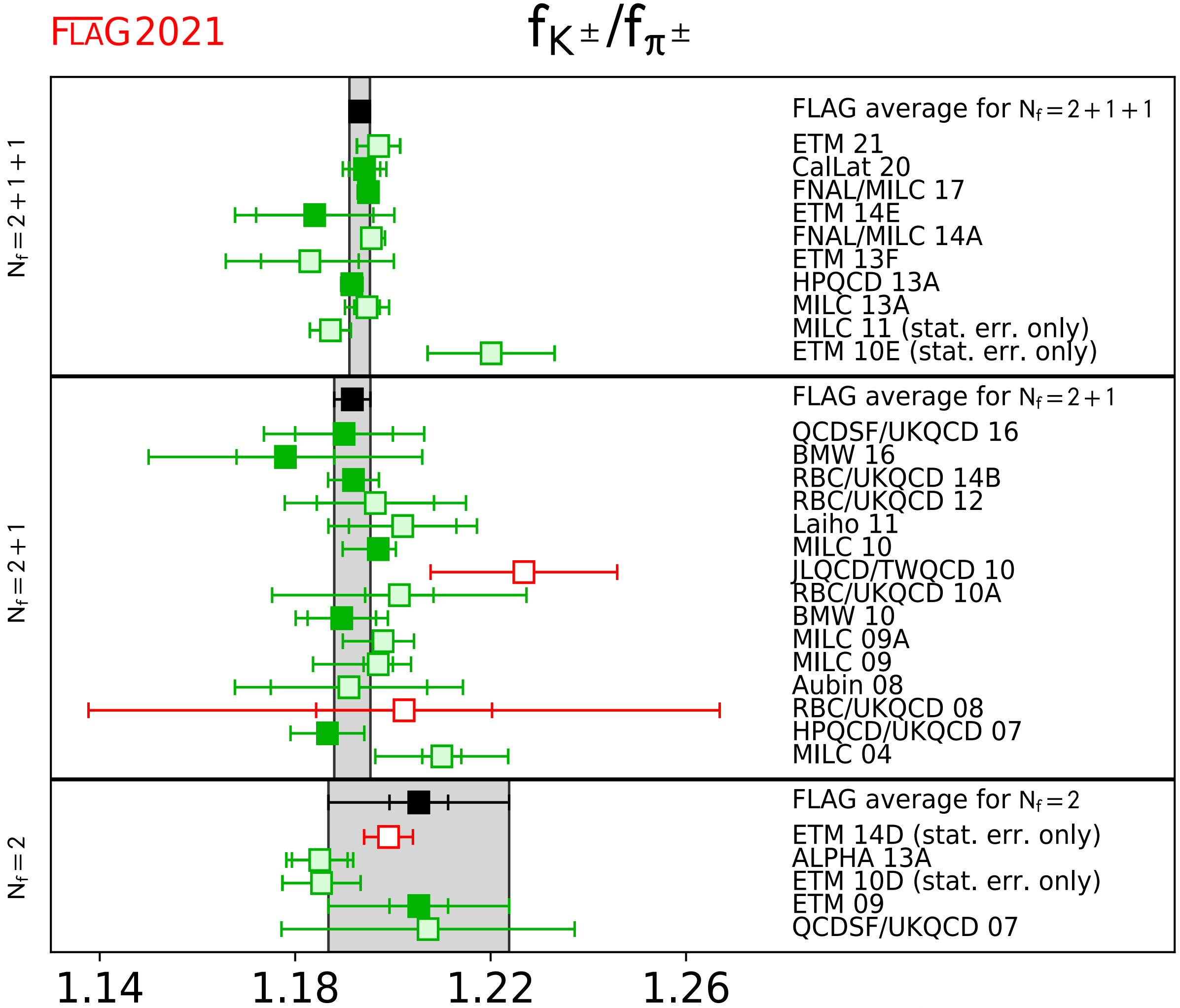
- In 2011: $V_{us}/V_{ud} = 0.2312(4)_{\mathrm{exp}}(12)_{\mathrm{lat}}$

- In 2021: $V_{us}/V_{ud} = 0.2311(3)_{\mathrm{exp}}(4)_{\mathrm{lat}}$ the lattice error is reducing by a factor of 3 compared to 2011! It is now of the same order as the experimental uncertainty.

-1.8σ away from unitarity

2.2 f_K/f_π from lattice QCD

Progress since 2018: ⇨ new results from *ETM'21* and *CalLat'20*



Now Lattice collaborations include SU(2) IB corr.

For N_f=2+1+1, FLAG2021

$$f_{K^+}\big/f_{\pi^+} = \mathbf{1.1932(21)}$$

0.18% uncertainty

Results have been stable over the years

For average substract IB corr.

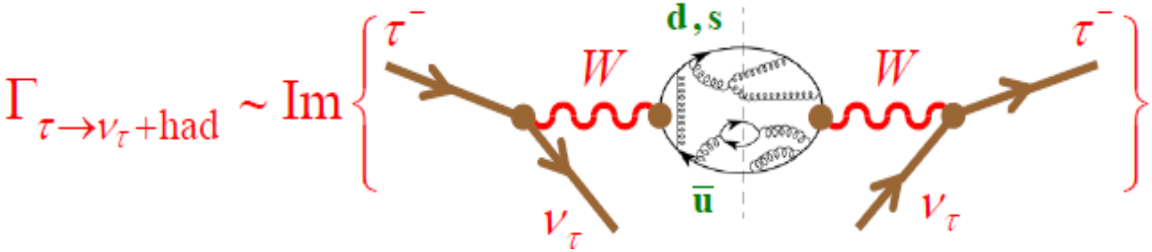
$$f_K\big/f_\pi = \mathbf{1.1967(18)}$$

In 2011: $f_K\big/f_\pi = \mathbf{1.193(6)}$

⇨ V_{us}/V_{ud} = 0. $23108(29)_{exp}(42)_{lat}$

Inclusive τ-decays

Braaten, Narison, Pich'92



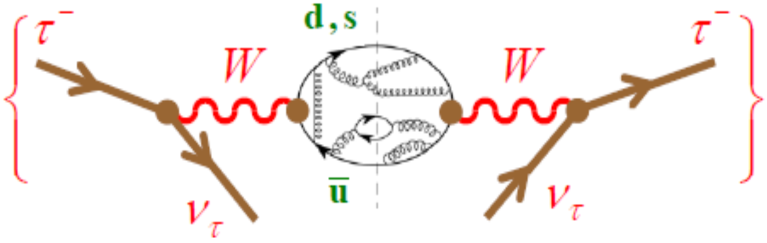
- Quantity of interest : $$\boxed{R_\tau \equiv \frac{\Gamma\left(\tau^- \to \nu_\tau + \textbf{hadrons}\right)}{\Gamma\left(\tau^- \to \nu_\tau e^- \bar{\nu}_e\right)}}$$

3.2 Calculation of the QCD corrections

Braaten, Narison, Pich'92

- Calculation of R_τ:

$$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}$$



$$\boxed{R_\tau(m_\tau^2) = 12\pi S_{EW}\int_0^{m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\text{Im}\,\Pi^{(1)}(s+i\varepsilon)+\text{Im}\,\Pi^{(0)}(s+i\varepsilon)\right]}$$

$$\Pi^{(J)}(s) \equiv |V_{ud}|^2\left(\Pi^{(J)}_{ud,V}(s)+\Pi^{(J)}_{ud,A}(s)\right)+|V_{us}|^2\left(\Pi^{(J)}_{us,V}(s)+\Pi^{(J)}_{us,A}(s)\right)$$

J

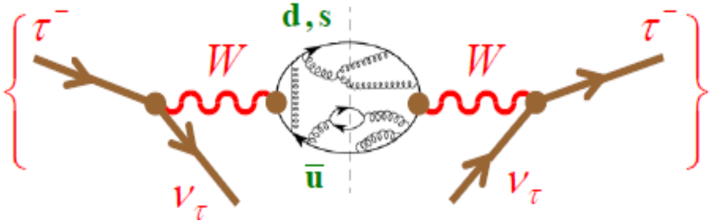
$$\Pi^{\mu\nu}_{ij,V/A}(q) = \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\Pi^{(1)}_{ij,V/A}(q^2) + q^\mu q^\nu\,\Pi^{(0)}_{ij,V/A}(q^2)$$

3.2 Calculation of the QCD corrections

Braaten, Narison, Pich'92

- Calculation of R_τ:

$$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}\left\{\quad\right\}$$



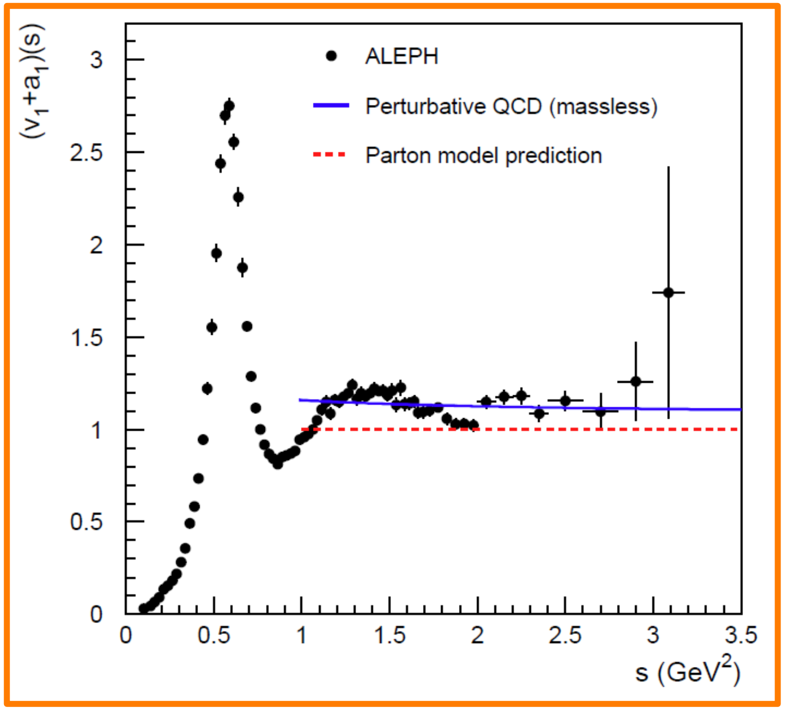
$$\Rightarrow\quad \boxed{R_\tau(m_\tau^2) = 12\pi S_{EW}\int_0^{m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\text{Im}\,\Pi^{(1)}(s+i\varepsilon)+\text{Im}\,\Pi^{(0)}(s+i\varepsilon)\right]}$$

- Spectral functions:

$$\text{Im}\Pi^{(1)}_{\bar{u}d,V/A}(s) = \frac{1}{2\pi}v_1/a_1(s)$$

- ALEPH and OPAL at LEP measured with precision not only the total BRs but also the energy distribution of the hadronic system

$\Rightarrow$ mix of non-perturbative and perturbative effects



Measurements

- $$\boxed{R_\tau \equiv \frac{\Gamma\left(\tau^- \to \nu_\tau + \text{hadrons}\right)}{\Gamma\left(\tau^- \to \nu_\tau e^- \bar{\nu}_e\right)} = ?}$$

- Decomposition as a function of observed and separated final states:

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

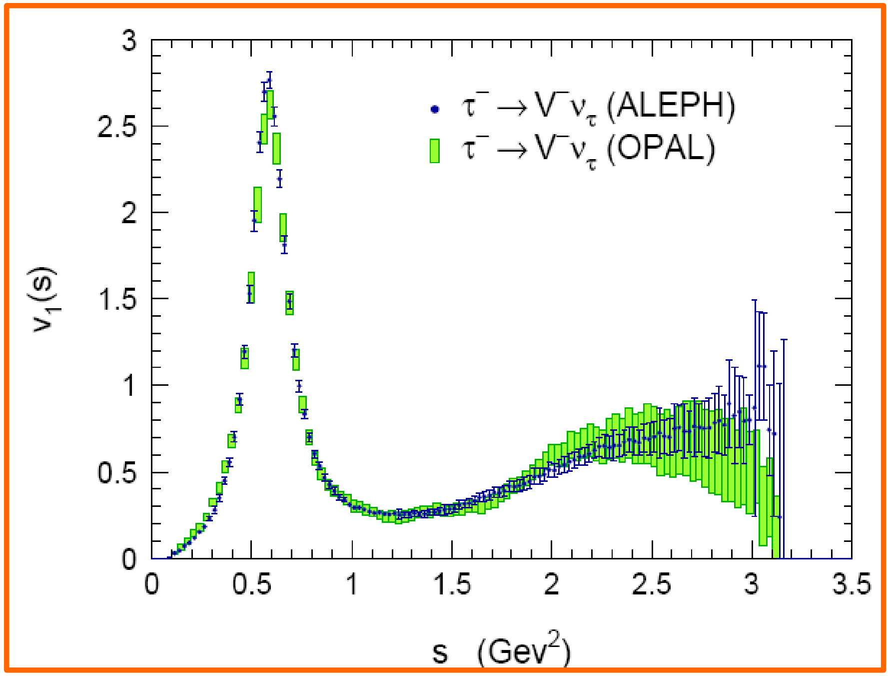
$$R_{\tau,V} \Rightarrow \boxed{\tau^- \to \nu_\tau + h_{v,s=0}}$$

(even number of pions)

$$R_{\tau,A} \Rightarrow \tau^- \to \nu_\tau + h_{A,s=0}$$

(odd number of pions)

$$R_{\tau,S} \Rightarrow \tau^- \to \nu_\tau + h_{V+A,s=1}$$



Measurements

- $$\boxed{R_\tau \equiv \frac{\Gamma\left(\tau^- \to \nu_\tau + \text{hadrons}\right)}{\Gamma\left(\tau^- \to \nu_\tau e^- \bar{\nu}_e\right)} = ?}$$

- Decomposition as a function of observed and separated final states:

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

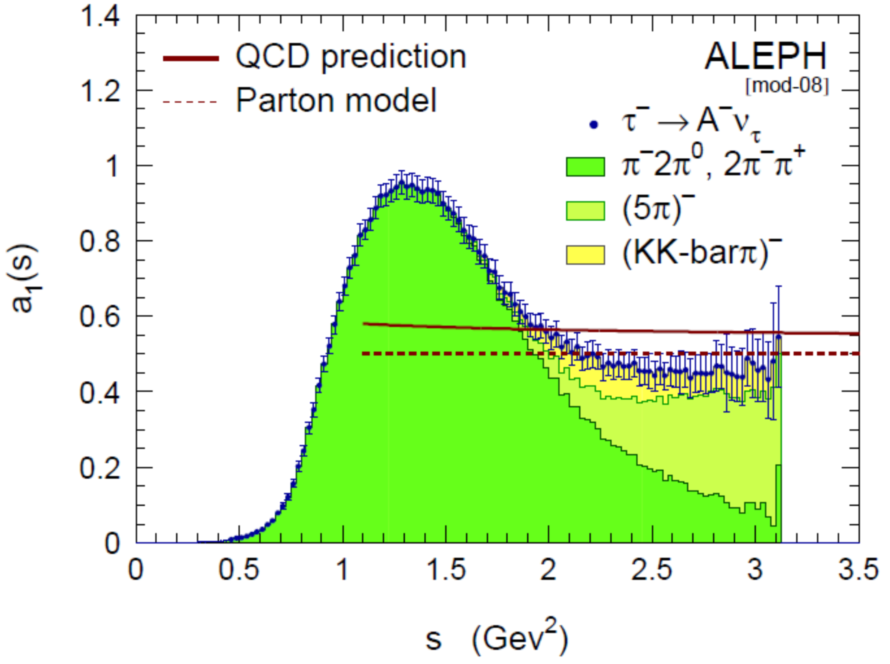
$R_{\tau,V}$ ⇨ $\tau^- \to \nu_\tau + h_{v,s=0}$

(even number of pions)

$R_{\tau,A}$ ⇨ $\boxed{\tau^- \to \nu_\tau + h_{A,s=0}}$

(odd number of pions)

$R_{\tau,S}$ ⇨ $\tau^- \to \nu_\tau + h_{V+A,s=1}$



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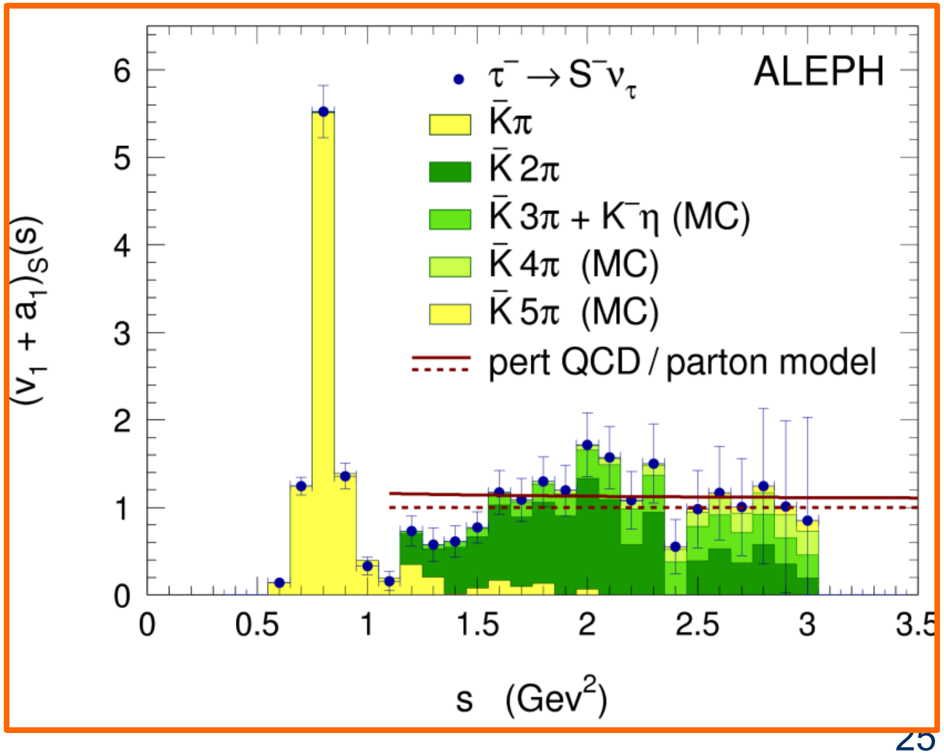
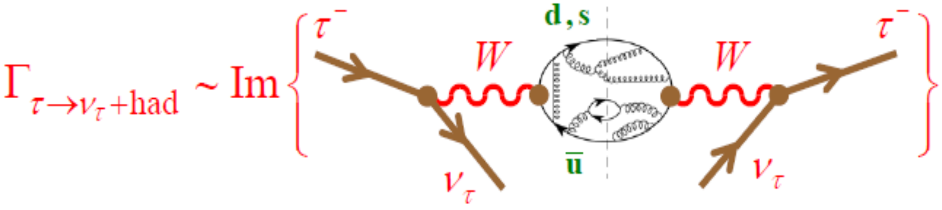


3.2 Calculation of the QCD corrections

Braaten, Narison, Pich'92

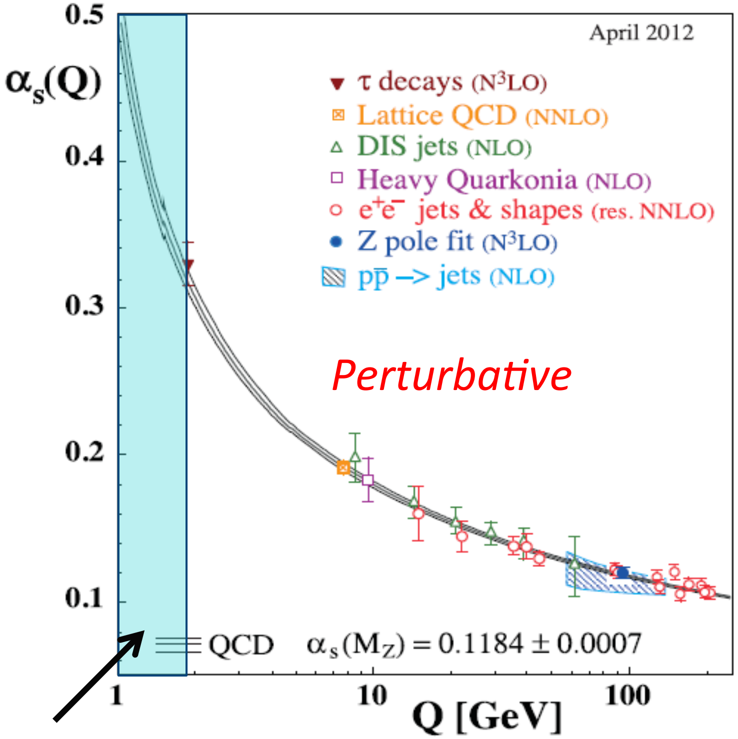
- Calculation of R_τ:



$$\boxed{R_\tau(m_\tau^2) = 12\pi S_{EW} \int_0^{m_\tau^2} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2 \left[\left(1+2\frac{s}{m_\tau^2}\right)\mathrm{Im}\,\Pi^{(1)}(s+i\varepsilon) + \mathrm{Im}\,\Pi^{(0)}(s+i\varepsilon)\right]}$$

- We are in the *non-perturbative* region: we do not know how to compute!

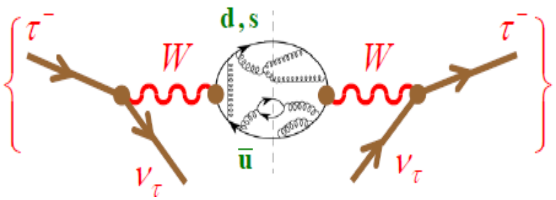
- Trick: use the analytical properties of Π!



Non-Perturbative

3.2 Calculation of the QCD corrections

$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}$



- Calculation of R_τ:

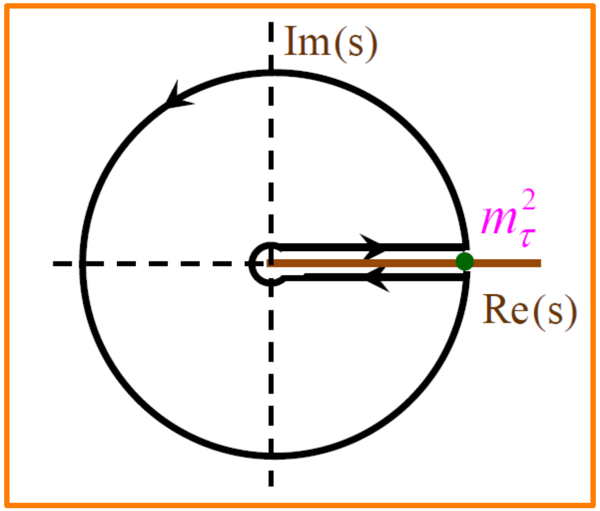
$$R_\tau(m_\tau^2)=12\pi S_{EW}\int_0^{m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\text{Im}\,\Pi^{(1)}(s+i\varepsilon)+\text{Im}\,\Pi^{(0)}(s+i\varepsilon)\right]$$

Braaten, Narison, Pich'92

- Analyticity: Π is analytic in the entire complex plane except for s real positive

⇨ Cauchy Theorem

$$R_\tau(m_\tau^2)=6i\pi S_{EW}\oint_{|s|=m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(1)}(s)+\Pi^{(0)}(s)\right]$$



- We are now at sufficient energy to use OPE:

$$\Pi^{(J)}(s)=\sum_{D=0,2,4\ldots}\frac{1}{(-s)^{D/2}}\sum_{\dim O=D}C^{(J)}(s,\mu)\,\langle O_D(\mu)\rangle$$

Wilson coefficients

Operators

μ: separation scale between short and long distances

3.3 Operator Product Expansion

$$\Pi^{(J)}(s) = \sum_{D=0,2,4\dots} \frac{1}{(-s)^{D/2}} \sum_{\dim O=D} \mathrm{C}^{(J)}(s,\mu)\,\langle O_D(\mu)\rangle$$

Wilson coefficients

Operators

μ separation scale between short and long distances

- D=0: Perturbative contributions
- D=2: Quark mass corrections
- D=4: Non perturbative physics operators, $\left\langle \frac{\alpha_s}{\pi} GG \right\rangle$, $\left\langle m_j \bar{q}_i q_i \right\rangle$
- D=6: 4 quarks operators, $\left\langle \bar{q}_i \Gamma_1 q_j \bar{q}_j \Gamma_2 q_i \right\rangle$
- D≥8: Neglected terms, supposed to be small…

$\Rightarrow$ $$R_{\tau,V}(s_0) = \frac{3}{2}\left|V^{ud}\right|^2 S_{EW}\left(1+\delta^{(0)} + \sum_{D=2,4..}\delta^{(D)}_{ud,V}\right)$$

similar for $R_{\tau,A}(s_0)$ and $R_{\tau,S}(s_0)$

Perturbative Part

Braaten, Narison, Pich'92

- Calculation of R_τ:

$$\boxed{R_\tau\left(m_\tau^2\right) = N_C\ S_{EW}\left(1+\delta_P+\delta_{NP}\right)}$$

- Electroweak corrections: $S_{EW} = 1.0201(3)$ *Marciano &Sirlin'88, Braaten & Li'90, Erler'04*

- Perturbative part (D=0):

$$a_\tau = \frac{\alpha_s(m_\tau)}{\pi}$$

$$\boxed{\delta_P = a_\tau + 5.20\ a_\tau^2 + 26\ a_\tau^3 + 127\ a_\tau^4 + \ldots \approx 20\%}$$

Baikov, Chetyrkin, Kühn'08

Non-perturbative part

Braaten, Narison, Pich'92

- Calculation of R_τ:

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Baikov, Chetyrkin, Kühn'08

- D=2: quark mass corrections, *neglected* for R_τ^{NS} $\left(\propto m_u, m_d\right)$ but not for R_τ^S $\left(\propto m_s\right)$

- D ≥ 4: Non perturbative part, not known, *fitted from the data*

  ⇨ Use of weighted distributions

Ex: In the non-strange sector: $\boxed{\delta_{NP}^{NS} = -0.0064(13)}$ *Davier et al.'14*

Non-Perturbative part

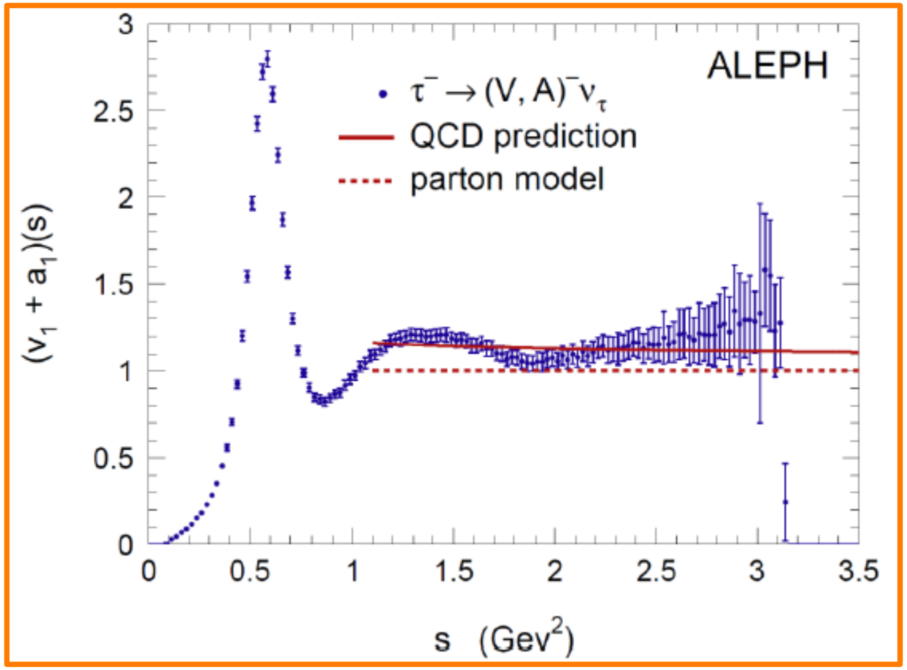
Le Diberder&Pich'92

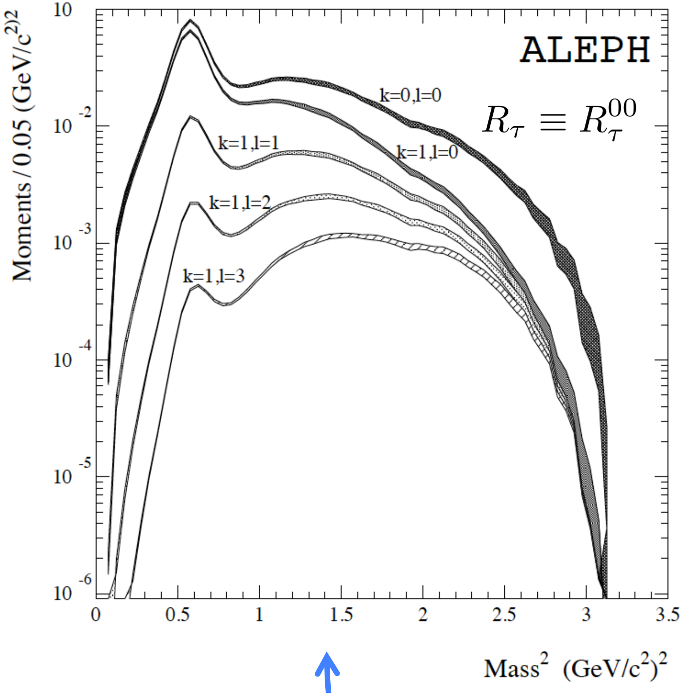
- D ≥ 4: Non perturbative part, not known, *fitted from the data*

  ⇨ Use of weighted distributions

Exploit shape of the spectral functions to obtain additional experimental information

$$R^{k\ell}_{\tau,U}(s_0) = \int_0^{s_0} ds \left(1-\frac{s}{s_0}\right)^k \left(\frac{s}{s_0}\right)^\ell \frac{dR_{\tau,U}(s_0)}{ds}$$





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