Unitarity Bounds for Hadronic Form Factors

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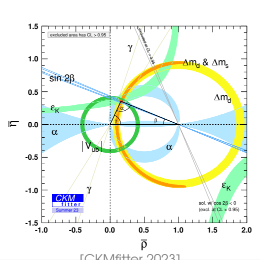
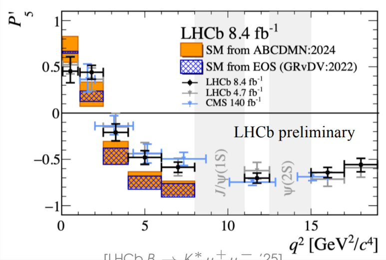
2025 Belle II Physics Week

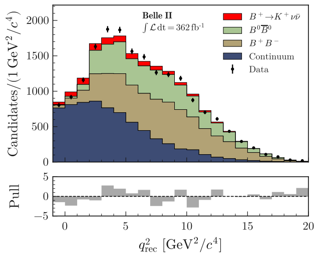
Need for Hadronic Form Factors



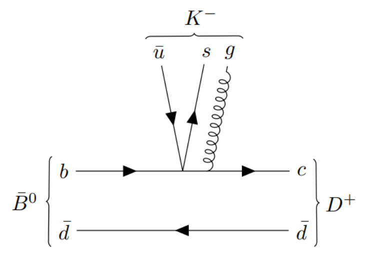
[CKMfitter 2023]



[LHCb $B \to K^* \mu^+ \mu^-$ '25]



[Belle II $B \to K\overline{\nu}\nu$ '23]



[Nonleptonic $\overline{B} \to DK$]

What is a Hadronic Form Factor?

Lorentz Decomposition (ex: $\overline{B} \to D\ell^-\overline{\nu}$ Form Factors)

$$\langle D(k)|\overline{c}\gamma^\mu b|\overline{B}(p)\rangle = \sum_n S_n^\mu(p,k)\; F_n(q^2 \equiv (p-k)^2)$$

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- describe mismatch between partonic current and hadronic transition
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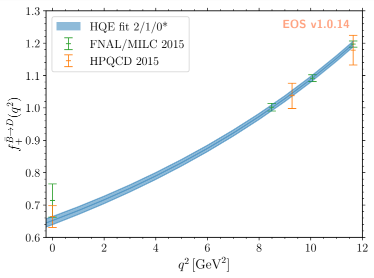

[Bordone,Gubernari,Jung,DvD '25]

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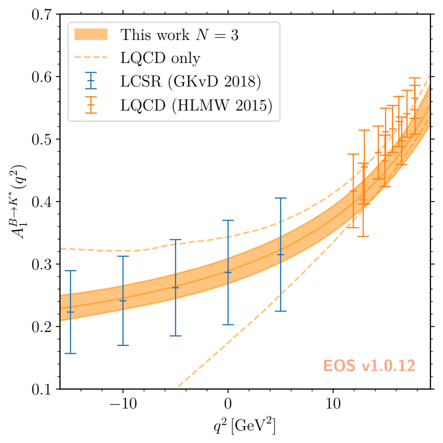

[Gubernari,Reboud,DvD,Virto '23]

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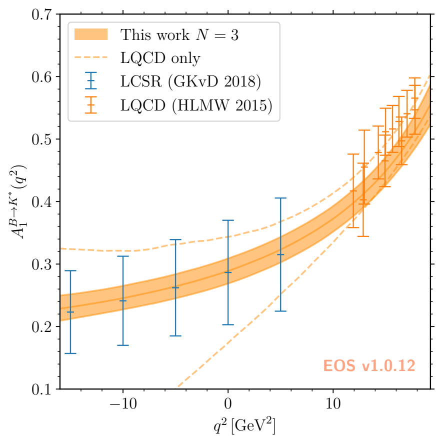

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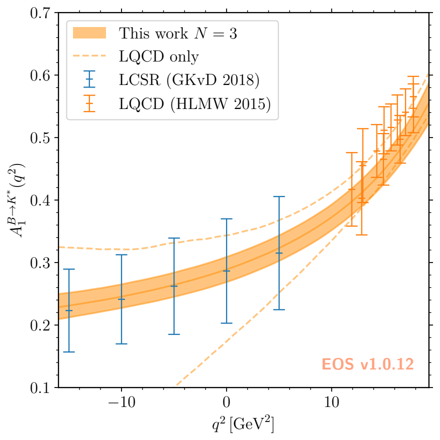
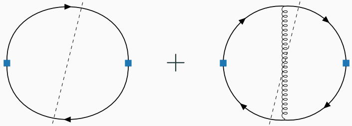


[Gubernari,Reboud,DvD,Virto '23]

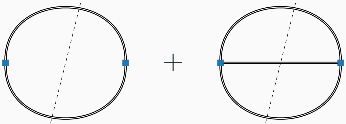
How to handle the systematic uncertainty inherent in the parametrization?

Enter Unitarity Bounds / Dispersive Bounds

$$\Pi(q^2) \equiv i \int d^4x e^{iq\cdot x} \langle 0|\mathcal{T}\{J^\dagger(x), J(0)\}|0\rangle$$



$+ \dots$ Disc Π



$+ \dots$

Partonic 2-point Function

- discontinuity accessible in operator product expansion

Hadronic 2-point Function

- discontinuity expressible in terms of hadronic form factors

Global Quark-Hadron Duality

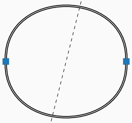
When summing & integrating over all on-shell intermediate states, the two discontinuities lead to the same result.

[Chibisov,Dikeman,Shifman,Uraltsev '96]

[Ling-Fong,Pagels '71] [Okubo '71] [Boyd,Grinstein,Lebed '94] [Caprini,Neubert '96]

[see also "Functional Analysis and Optimization Methods in Hadron Physics" (I. Caprini)]

Wait, what?



How does this relate to the form factors? $\langle BD|J|0\rangle$?

Crossing Symmetry

$$\langle D(k)|J|\overline{B}(+p)\rangle = \sum_n S_n^\mu(+p,k)\; F_n(q^2 \equiv (+p-k)^2)$$
$$\langle D(k)B(-p)|J|0\rangle = \sum_n S_n^\mu(-p,k)\; F_n(q^2 \equiv (-p-k)^2)$$

- the same function / form factor describes the semileptonic decay and the pair production for different regions of q^2
  - for $0 \leq q^2 \leq (M_B - M_D)^2$, it describes the semileptonic decay $\overline{B} \to D\ell^-\overline{\nu}$
  - for $(M_B + M_D)^2$, it describes the pair production $\ell^+\nu \to BD$
- form factors appear "squared"

Creating the Bound

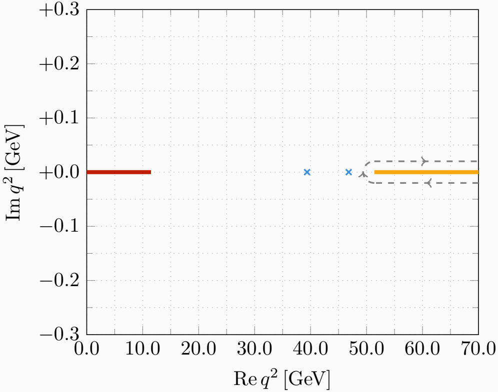
- Π is typically divergent, but its derivatives become finite
- define χ

$$\chi(Q^2) = \frac{d^n}{d(Q^2)^n}\Pi(Q^2) = \frac{1}{2\pi i}\frac{d^n}{d(Q^2)^n}\oint_C dt\,\frac{\Pi(t)}{t-Q^2}$$

  - select n as the smallest number of derivatives so that $\chi(Q^2)$ is finite
  - choose contour C to avoid singularities
- compute $\chi(Q^2=0)$ in an Operator Product Expansion (OPE)
- we can relate $\mathrm{Disc}\,\Pi$ to the form factors, but not Π to the formfactors!

Properties of Π in a Nutshell

$$\Pi(q^2) \equiv i \int d^4x e^{iq\cdot x} \, \langle 0|\mathcal{T}\{J^\dagger(x), J(0)\}|0\rangle$$



relevant here:

- pair production branch cut
- integration contour
- $\dfrac{\Pi(t)}{t^{n+1}} \to 0$ as $|t| \to \infty$

contour shrinks to two rays

- below the cut (moving left)
- above the cut (moving right)

$$\Pi(t + i\varepsilon) - \Pi(t - i\varepsilon) \equiv \text{Disc}\,\Pi(t + i\varepsilon)$$

Creating the Bound (2)

$$\chi^{\mathrm{FF}}(Q^2) = \frac{1}{2\pi i}\frac{d^n}{d(Q^2)^n}\oint_C dt\,\frac{\Pi^{\mathrm{FF}}(t)}{t-Q^2} = \frac{1}{2\pi i}\frac{d^n}{d(Q^2)^n}\int_{(M_B+M_D)^2}^{\infty} dt\,\frac{\mathrm{Disc}\,\Pi^{\mathrm{FF}}(t)}{t-Q^2}$$

- compute discontinuity of Π in terms of the hadronic form factors, single out one form factor F and its (positive) kinematic weight function

$$i\,\mathrm{Disc}\,\Pi^{\mathrm{FF}}(t) = \omega(t)|F(t)|^2 + \text{positive terms}$$

- drop positive terms to create inequality, the dispersive bound

$$\boxed{\chi^{\mathrm{OPE}}(0) \geq \frac{1}{2\pi}\frac{d^n}{d(Q^2)^n}\int dt\,\frac{\omega(t)|F(t)|^2}{t-Q^2}\Bigg|_{Q^2=0}}$$

Boyd-Grinstein-Lebed Idea

- create separate bounds for vector and axialvector currents
- can we create a parametrization that manifestly respects the bound?
- problematic quantity:

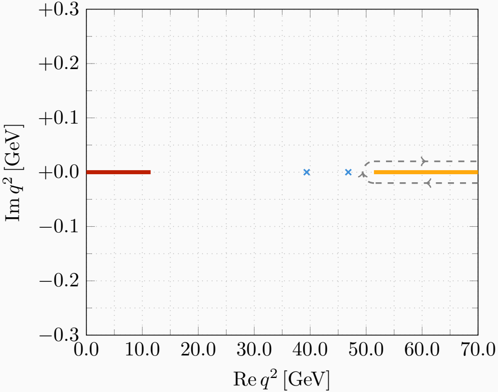
$$\frac{1}{2\pi}\frac{d^n}{d(Q^2)^n}\int_{(M_B+M_D)^2}^{\infty} dt\,\frac{\omega(t)|F(t)|^2}{t-Q^2}\bigg|_{Q^2=0} \overset{?}{\longrightarrow} \int_{(M_B+M_D)^2}^{\infty} dt\,|\tilde{F}(t)|^2$$

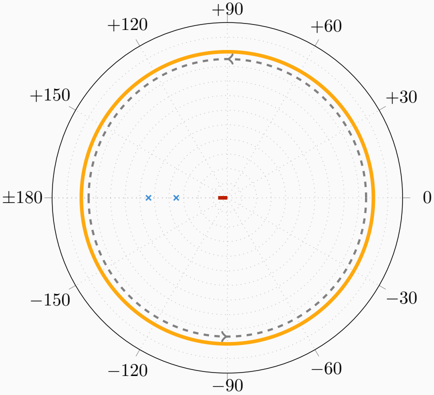
- can we "diagonalise" the contribution to the integral by expanding $\tilde{F}$ in a suitable basis of functions?

[Boyd,Grinstein,Lebed '95]

Mapping

$$t \mapsto z(t) \equiv \frac{\sqrt{(M_B+M_D)^2 - t} - \sqrt{(M_B+M_D)^2 - (M_B-M_D)^2}}{\sqrt{(M_B+M_D)^2 - t} - \sqrt{(M_B+M_D)^2 - (M_B-M_D)^2}}$$





$$\frac{1}{2\pi}\frac{d^n}{d(Q^2)^n}\int_{(M_B+M_D)^2}^{\infty} dt\,\frac{\omega(t)|F(t)|^2}{t-Q^2}\bigg|_{Q^2=0} = \int_{-\pi}^{+\pi}\frac{d\vartheta}{2\pi}\, z J_{z\to t}(z)\left[\frac{d^n}{d(Q^2)^n}\frac{\omega(t)|F(t)|^2}{t-Q^2}\right]_{z=e^{i\vartheta},Q^2=0}$$

Boyd-Grinstein-Lebed Parametrization

- absorb χ, weight factor ω, and kernel $1/(t - Q^2)$ into "outer function" ϕ

$$1 \geq \frac{1}{\chi(0)} \int_{-\pi}^{+\pi} \frac{d\vartheta}{2\pi} z\, J_{z\to t}(z) \left[\frac{d^n}{d(Q^2)^n} \frac{\omega(t)|F(t(z))|^2}{t - Q^2}\right]_{z=e^{i\vartheta}, Q^2=0} \equiv \int_{-\pi}^{+\pi} \frac{d\vartheta}{2\pi} \left|\phi(z)F(t(z))\right|^2 \Big|_{z=e^{i\vartheta}}$$

- use an orthonormal basis of analytic functions on the unit circle

$$\int_{-\pi}^{+\pi} \frac{d\vartheta}{2\pi} z^k z^{*,l} = \delta_{kl} \qquad \text{for } k, l \geq 0$$

- expand $\phi(z)F(t(z))$ into a series around $z = 0$

$$F(t) = \frac{1}{\phi(z(t))} \sum_{k=0}^{\infty} a_k [z(t)]^k \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} |a_k|^2 \leq 1$$

- each expansion coefficient is absolutely bounded to the interval $[-1, +1]$

Practical Considerations

- truncate series at order K

$$F_K(t) = \frac{1}{\phi(z(t))} \sum_{k=0}^{K} a_k [z(t)]^k$$

- what is the truncation error?

$$\begin{aligned}
\epsilon_K(z) \equiv \left| \phi(z)\left[F(t(z)) - F_K(t(z))\right] \right| &= \left| \sum_{k=K+1}^{\infty} a_k z^k \right| \\
&\leq \left\{ \left[\sum_{k=K+1}^{\infty} |a_k|^2 \right] \left[\sum_{k=K+1}^{\infty} |z_k|^2 \right] \right\}^{1/2} \\
&= \sqrt{A_K \cdot Z_K(z)}
\end{aligned}$$

- identify $A_K \leq 1 - \sum_{k=0}^{K} |a_k|^2 < 1$ and $Z_K(z) \leq |z^2|^{K+1} \sum_{k=0}^{\infty} |z^2|^k = \frac{|z^2|^{K+1}}{1-|z^2|}$

$$\epsilon_K(z) < \text{constant} \times |z|^{K+1}$$

Further Literature / Not Covered

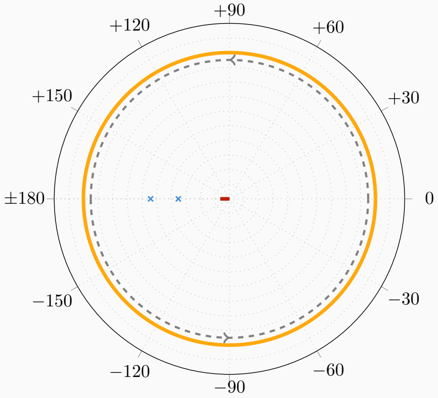
- subthreshold branch cuts / multiple branch cuts [Blake,Meinel,Rahimi,DvD '22] [Flynn,Jüttner,Tsang '23]
- including resonances / working on the branch cut [Buck,Lebed '98] [Kirk,Kubis,Reboud,DvD '24]
- nonlocal form factors ("charm loop" in $\overline{B} \to \overline{K}\mu^+\mu^-$) [Gubernari,DvD,Virto '20] [Gubernari,Reboud,DvD,Virto '22]
- dispersive matrix / avoiding truncation [Okubo '71] [Lellouch '95] [Di Carlo,Martinelli,Naviglio,Sanfilippo,Simula,Vittorio '21]

Summary

- hadronic form factors are relevant to a large number of processes relevant to Belle II measurements (and beyond!)
- control of systematic uncertainties in hadronic form factors crucial to interpretation of measurements
- dispersive bounds provide excellent tool to control these uncertainties
    - in simple cases (as in the example shown here), dispersive bounds provide strict upper bound on expansion coefficients
- active field of research, revitalized in recent years with applications beyond what was originally envisaged

Backup Slides

What about the Poles?



- e.g. for vector form factor

$$B_c^*, B_c^*(6842), \ldots$$

- F receives contributions form one-shell intermediate 1-body states
- produce poles in complex z plane
- can we remove them without changing the integral on the circle?
- replace $\phi F \to B\phi F$ before expansion, where

$$B(z = z_{\text{pole}}) = 0$$

$$\left|B(z = e^{i\vartheta})\right| = 1$$

- solutions for B are called "Blaschke factors"