Amplitude Analyses

BELLE II Physics Week, 30 Nov 2020
(originally for B Workshop, Neckarzimmern, 2015)
Jonas Rademacker

Special thanks to Antimo Palano, Marco Pagapallo, and Patricia Magalhães, from whose excellent talks I lifted a particularly large number of plots.
Why Amplitude Analyses?

• QM is intrinsically complex:

  Wave functions/transition amplitudes etc: \( \psi = a e^{i\alpha} \). Observable: \( |\psi|^2 \).

  Only half the information. How do I get the rest?

• Note that the rest is very interesting - CP violation in the SM comes from phases!

• Answer: Interference effects:

\[
\psi_{\text{total}} = a e^{i\alpha} + b e^{i\beta} + \ldots \\
|\psi_{\text{total}}|^2 = |a e^{i\alpha} + b e^{i\beta} + \ldots|^2 = a^2 + b^2 + 2ab \cos(\alpha - \beta) + \ldots
\]
Dalitz plot analyses - lots of interfering amplitudes!

Many interfering decay paths contribute to the same final state:

\[ K^*(892)^+\pi^- \]
\[ K^*(892)^-\pi^+ \]
\[ K_0^*(1430)^+\pi^- \]
\[ K_0^*(1430)^-\pi^+ \]
\[ K_2^*(1430)^+\pi^- \]
\[ K_2^*(1430)^-\pi^+ \]
\[ K^*(1680)^+\pi^- \]
\[ K^*(1680)^-\pi^+ \]
\[ K_S\rho^0 \]
\[ K_S\omega \]
\[ K_Sf_0(980) \]
\[ K_Sf_0(1370) \]
\[ K_Sf_2(1270) \]
\[ K_S\sigma_1 \]
\[ K_S\sigma_2 \]
non-resonant

Described by a sum of complex amplitudes:

\[ A(s_+, s_-) = \sum_k a_k(s_+, s_-) e^{i\phi_k(s_+, s_-)} \]

\[ |A(s_+, s_-)|^2 \]

represented in a Dalitz plot.
3 body decays

\[ d\Gamma = |M_{fi}|^2 d\Phi \]
\[ = |M_{fi}|^2 \left| \frac{\partial \Phi}{\partial (s_{12}, s_{13})} \right| ds_{12} ds_{13} \]
\[ = \frac{1}{(2\pi)^2 32M^3} |M_{fi}|^2 ds_{12} ds_{13} \]

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
3-body phase space

$s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2$
3-body phase space

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
3-body phase space

\[
S_{13} \quad (m_1 + m_2)^2 \quad (M - m_3)^2 \quad S_{12}
\]

\[
s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2
\]
3-body phase space

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
3-body phase space

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
3-body phase space

\[ M^2 + m_1^2 + m_2^2 + m_3^2 = s_{12} + s_{13} + s_{23} \]

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
3-body phase space

\[ M^2 + m_1^2 + m_2^2 + m_3^2 = s_{12} + s_{13} + s_{23} \]

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
3-body phase space

\[ M^2 - m_1^2 + 2m_2m_3 \]
\[ m_2^2 + m_3^2 + Mm_1 \]
\[ (m_1 + m_3)^2 \]
\[ M^2 + m_1^2 + m_2^2 + m_3^2 = s_{12} + s_{13} + s_{23} \]

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
3-body phase space

\[ M^2 - m_1^2 + 2m_2m_3 = (M - m_1)^2 \]

\[ m_2^2 + m_3^2 + Mm_1 = (m_1 + m_3)^2 \]

\[ M^2 + m_1^2 + m_2^2 + m_3^2 = s_{12} + s_{13} + s_{23} \]

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
3-body phase space

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]
What happens if nothing happens

\[ M_{fi} = 1 \]

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |M_{fi}|^2 ds_{12}ds_{13} \]
What really happens

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |M_{fi}|^2 ds_{12} ds_{13} \]
What happens if one thing happens

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |M_{fi}|^2 ds_{12} ds_{13} \]
What happens if one thing happens

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |M_{fi}|^2 ds_{12}ds_{13} \]
What happens if one thing happens

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |\mathcal{M}_{fi}|^2 ds_{12} ds_{13} \]
What happens if two things happens

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |\mathcal{M}_{fi}|^2 ds_{12}ds_{13} \]
What happens if two things happens

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |\mathcal{M}_{fi}|^2 ds_{12} ds_{13} \]
What happens if something with spin happens

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |\mathcal{M}_{fi}|^2 ds_{12}ds_{13} \]
Why you must do an amplitude analyses if you want to find real new resonances in multi body decays.

\[ D^0 \rightarrow K_S \pi \pi \]

\( \pi \pi \) resonance near \( m^2 = 2 \text{GeV}^2 \)?

CDF PHYSICAL REVIEW D 86, 032007 (2012) (no claim of any bogus resonance is made in this paper, it's a completely sound paper about CPV in charm).
Why you must do an amplitude analyses if you want to find \textit{real} new resonances in multi body decays.

$D^0 \rightarrow K_{S\pi\pi}$

CDF PHYSICAL REVIEW D 86, 032007 (2012) (no claim of any bogus resonance is made in this paper, it’s a completely sound paper about CPV in charm).

Structure due to angular distribution in $D \rightarrow K^*(K_{S\pi})\pi$

\textbf{Not a (new or old) } $\pi\pi$ resonance
Real Dalitz plots

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |M_{fi}|^2 ds_{12} ds_{13} \]
Real Dalitz pots

\[ s_{ij} \equiv (p_i + p_j)^2 \equiv m_{ij}^2 \]

\[ d\Gamma = \frac{1}{(2\pi)^2 32M^3} |\mathcal{M}_{fi}|^2 ds_{12} ds_{13} \]

\[ 2.4M \, D^{\pm} \to \pi^\pm \pi^\mp \pi^\pm \text{ decays (LHCb)} \]

\[ \sigma(500)? \]

\[ \rho(770) \to f(980) \]

\[ \text{LHCb} \]

\[ \text{Phys. Lett. B728 (2014) 585} \]
Calculating amplitudes

- Let us assume(!) that the full amplitude can be calculated as the sum of essentially independent two-body processes.

- Doing this results in the so-called “isobar” model.
Calculating amplitudes

- We don’t know anything about the strong interaction dynamics.
- As a first approximation, we treat each particle as point particle.
- We want a Lorentz-invariant matrix element...
Calculating amplitudes

\[ \frac{1}{s_{23} - m_R^2 - i m_R \Gamma} \]

- Diagram showing particle interactions.
- Graphs illustrating various functions and data points.

Jonas Rademacker: Amplitude Analyses
B-workshop
Neckarzimmern 18 Feb 2015
Calculating the amplitudes

\( \varepsilon_R \) say \( R \) has spin 1 (e.g. \( K^*(892) \), \( \rho(770) \) etc)

\( q_{23}^\nu \equiv p_2^\nu - p_3^\nu \)

\( p_1^\mu \)

M \hline

R \hline

3 \hline

\( \varepsilon_R \)

2 \hline

\( q_{23}^\nu \equiv p_2^\nu - p_3^\nu \)

1 \hline

\( p_1^\mu \)
Calculating the amplitudes

\[ \varepsilon_R \]

\[ q_{23}^{\nu} \equiv p_2^{\nu} - p_3^{\nu} \]

\[ p_1^\mu \]

\[ m_R \]

\[ \frac{1}{s_{23} - m_R^2 - i m_R \Gamma} \]
Calculating the amplitudes

\[ \varepsilon_R \]
\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]
\[ p_1^\mu \]

\[ p_1^\mu \varepsilon_R^{\nu*} \frac{1}{s_{23} - m_R^2 - i m_R \Gamma} \varepsilon_R^\nu \qquad q_{23} \nu \]
Calculating the amplitudes

\[ \varepsilon_R^\sigma \]

\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

\[ M \]

\[ \sum_{\text{all } \lambda} \frac{p_1^\mu \varepsilon_R^{\lambda \mu*}}{s_{23} - m_R^2 - i m_R \Gamma} \frac{1}{\varepsilon_R^{\lambda \nu}} \]

\[ q_{23}^\nu \]
Calculating the amplitudes

\[ \varepsilon_R^\sigma \]

\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

\[ \sum_{\text{all } \lambda} \varepsilon_R^{\lambda \mu*} \varepsilon_R^{\lambda \nu} = -g^{\mu \nu} + \frac{p_R^\mu p_R^\nu}{p_R^2} \]

\[ \sum_{\text{all } \lambda} p_1^\mu \varepsilon_R^{\lambda \mu*} \frac{1}{s_{23} - m_R^2 - im_R \Gamma} \varepsilon_R^{\lambda \nu} q_{23}^\nu \]
Calculating the amplitudes

\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

\[ \varepsilon_R^\sigma \]

\[ M \rightarrow R \]

\[ p_1^\mu \]

\[ q_{23}^\nu \]

\[ \varepsilon_R^\sigma \]

\[ \kappa^* \]

\[ \varepsilon_R^\sigma \]

\[ p_1^\mu \]

\[ \frac{-g^{\mu\nu} + \frac{p_R^\mu p_R^\nu}{p_R^2}}{s_{23} - m_R^2 - i m_R \Gamma} \]

\[ q_{23}^\nu \]
Calculating the amplitudes

$$\varepsilon_R$$

$$q_{23}^\nu \equiv p_2^\nu - p_3^\nu$$

$$p_1^\mu$$

M

R

3

2

1

$$\varepsilon_R$$

$$q_{23}^\nu$$

$$\varepsilon_R$$

$$p_1^\mu$$

spin factor

$$p_{1\mu} \left( -g^{\mu\nu} + \frac{p_R^{\mu} p_R^{\nu}}{p_R^2} \right) \frac{s_{23} - m_R^2 - i m_R \Gamma}{q_{23}^\nu}$$
Calculating the amplitudes

\[ \varepsilon_R^\sigma \]

\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

\[ R \]

\[ M \]

\[ -p_1 \cdot q_{23} + \frac{(p_1 \cdot p_R)(q_{23} \cdot p_R)}{p_R^2} \]

\[ \frac{1}{s_{23} - m_R^2 - i m_R \Gamma} \]

spin factor (here for L=1)
Calculating the amplitudes

\[ \varepsilon_R^\sigma \]

\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

Express in terms of \( s_{ij} \) if you wish, using \( p_i \cdot p_j = s_{ij} - m_i^2 - m_j^2 \)

\[ \left( -p_1 \cdot q_{23} + \frac{(p_1 \cdot p_R)(q_{23} \cdot p_R)}{p_R^2} \right) \frac{1}{s_{23} - m_R^2 - i m_R \Gamma} \]

spin factor (here for \( L=1 \))
Calculating the amplitudes

\[ \varepsilon_R^\sigma \]

\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

\[ \begin{align*}
    p_1^\mu & \left( -g^{\mu\nu} + \frac{p_R^\mu p_R^\nu}{p_R^2} \right) \\
    & \frac{s_{23} - m_R^2 - i m_R \Gamma}{q_{23}^\nu}
\end{align*} \]
Calculating the amplitudes

Angular Momenta require momenta

\[ q_{23} \equiv p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

\[ \varepsilon_\sigma_R \]

\[ R \text{ (with } J=1) \]

\[ M \]

\[ l=1 \]

\[ 1 \]

\[ 2 \]

\[ 3 \]

\[ \vec{q}_r \]

\[ \vec{L} = 2 \vec{d} \times \vec{q}_r \]

\[ L = \sqrt{l(l+1)} \]

\[ p_1^\mu \frac{-g^{\mu\nu} + p_1^\mu p_1^\nu}{p_R^2 - m_R^2 - i m_R \Gamma} q_{23}^\nu \]

in decay rest frame

classical mechanics

QM
Blatt Weisskopf Penetration Factors

<table>
<thead>
<tr>
<th>$L$</th>
<th>$B_L(q)$</th>
<th>$B'_L(q, q_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\sqrt{\frac{2z}{1+z}}$</td>
<td>$\sqrt{\frac{1+z_0}{1+z}}$</td>
</tr>
<tr>
<td>2</td>
<td>$\sqrt{\frac{13z^2}{(z-3)^2+9z}}$</td>
<td>$\sqrt{\frac{(z_0-3)^2+9z_0}{(z-3)^2+9z}}$</td>
</tr>
</tbody>
</table>

where $z = (|q|d)^2$ and $z_0 = (|q_0|d)^2$

Classical mechanics:
$L = 2qd$

QM:
$L^2 = l(l+1)$
Blatt Weisskopf Penetration Factors

\[
\frac{1}{4} L^2 = \frac{1}{4} l(l+1) \text{ for } l = \ldots 1 \ 2 \ 3 \ 4
\]

classical mechanics:
\[ L = 2 qd \]

QM:
\[ L^2 = l(l+1) \]
Calculating the amplitudes

Angular Momenta require momenta

\[ \varepsilon^\sigma_R \]

\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

\[ R \text{ (with J=1)} \]

\[ \langle q_{rM}, d_M \rangle \]

\[ \frac{-g^{\mu\nu} + \frac{p_R^\mu p_R^\nu}{p_R^2}}{s_{23} - m_R^2 - i m_R \Gamma} \]

\[ B'_L(q_{rR}, d_R) \]

\[ q_{23} \nu \]
Calculating the amplitudes

- Width $\Gamma = r$ rate, depends on phase space $= 2q/m$.
- Rate also depends on $B_L$.

\[ p_{1\mu} B_L(q_R, d_M) \frac{-g^{\mu\nu} + \frac{p_R^\mu p_R^\nu}{p_R^2}}{s_{23} - m_R^2 - im_R\Gamma} B_L(q_R, d_R) q_{23} \nu \]
Calculating the amplitudes

\[ \varepsilon_R^\sigma \]

\[ q_{23}^\rho \equiv p_2^\rho - p_3^\rho \]

\[ p_1^\mu \]

\[ R \] (with \( J=1 \))

\[ l=1 \]

\[ 3 \]

\[ 2 \]

\[ 1 \]

- Width \( \Gamma = \text{rate} \), depends on phase space = \( 2q/m \).

- Rate also depends on \( B_L \).

\[ \Gamma = \text{rate, depends on phase space} = 2q/m. \]

\[ B_L(q_{rR}, d_R) \] break-up momentum

\[ J = 1 \]

\[ p_1^\mu \ B_L(q_{rM}, d_M) \]

\[ \begin{align*}
- g^{\mu\nu} + \frac{p_R^\mu p_R^\nu}{p_R^2} & \frac{B_L(q_{rR}, d_R) q_{23}^\nu}{s_{23} - m_R^2 - i m_R \Gamma(m_{23})} \\
\end{align*} \]
Calculating the amplitudes

\[ \varepsilon_R \]

\[ q_{23} = p_2^\nu - p_3^\nu \]

\[ p_1^\mu \]

- Width \( \Gamma = \) rate, depends on phase space = \( 2q/m \).

- Rate also depends on \( B_L \).

\[ \Gamma(m_{23}) = \Gamma_0 \frac{(q_{23}/m_{23}) B_L(q_{23})}{(q_0/m_R) B_L(q_0)} \]

\[ p_1^\mu B_L(q_{rM}, d_M) \frac{-g^{\mu\nu} + \frac{p_R^\mu p_R^\nu}{p_R^2}}{s_{23} - m_R^2 - i m_R \Gamma(m_{23})} B_L(q_{rR}, d_R) q_{23}^\nu \]

\( J = 1 \)
Calculating the amplitudes

- Width $\Gamma = \text{rate}$, depends on phase space $= 2q/m$.
- Rate also depends on $B_L$.

$\varepsilon_R$
$q_{23}^\nu \equiv p_2^\nu - p_3^\nu$
$p_1^\mu$

$M \rightarrow R (\text{with } J = 1)$

$|l| = 1$

$\varepsilon_R$ break-up momentum in restframe of decaying resonance

$\Gamma(m_{23}) = \Gamma_0 \frac{(q_{23}/m_{23}) B_L(q_{23})}{(q_0/m_R) B_L(q_0)}$

$|l| = 1$

reconstructed mass $m_{23} \equiv \sqrt{s_{23}}$

break-up momentum

$\frac{p_1 \mu B_L(q_{rM}, d_M)}{s_{23} - m_R^2 - i m_R \Gamma(m_{23})} - \frac{p_{R}^\mu p_R^\nu}{p_R^2} \frac{B_L(q_{rR}, d_R)}{q_{23} \nu} J = 1$
Calculating the amplitudes

- Width $\Gamma = \text{rate, depends on phase space} = 2q/m$.
- Rate also depends on $B_L$.

\[ \Gamma(m_{23}) = \Gamma_0 \frac{(q_{23}/m_{23}) B_L(q_{23})}{(q_0/m_R) B_L(q_0)} \]

\[ p_1 \mu B_L(q_{RM}, d_M) \frac{-g^{\mu\nu} + \frac{p_R^\mu p_R^\nu}{p_R^2}}{s_{23} - m_R^2 - i m_R \Gamma(m_{23})} B_L(q_{Rd}, d_R) q_{23} \nu \]

break-up momentum
reconstructed mass $m_{23} \equiv \sqrt{s_{23}}$
centrifugal barrier factor
break-up momentum in restframe of decaying resonance
Calculating the amplitudes

- Width $\Gamma = \text{rate}$, depends on phase space $= 2q/m$.
- Rate also depends on $B_L$.

\[ \varepsilon_R^{\sigma} \]
\[ q_{23}^\nu \equiv p_2^\nu - p_3^\nu \]
\[ p_1^\mu \]

$R \ (\text{with } J=1)$

\[ l=1 \]

\[ l=1 \]

\[ l=1 \]

\[ \text{break-up momentum in restframe of decaying resonance} \]
\[ \text{the same as numerator, but calculated for “nominal” (peak) resonance mass.} \]

\[ \Gamma(m_{23}) = \Gamma_0 \left( \frac{q_{23}/m_{23}}{q_0/m_R} \right) \frac{B_L(q_{23})}{B_L(q_0)} \]

\[ p_1^\mu B_L(q_{rM}, d_M) \]

\[ -g^{\mu \nu} + \frac{p_R^\mu p_R^\nu}{p_R^2} \]

\[ \frac{s_{23} - m_R^2 - im_R \Gamma(m_{23})}{B_L(q_{rR}, d_R)} q_{23}^\nu \]
Mass dependent width (ignoring ang. mom)

\[ \Gamma(m_{23}) = \Gamma_0 \frac{(q_{23}/m_{23})}{(q_0/m_R)} \frac{B_L(q_{23})}{B_L(q_0)} \]
Breit Wigner with angular momentum effects (only)

\[ \Gamma(m_{23}) = \Gamma_0 \frac{(q_{23}/m_{23}) B_L(q_{23})}{(q_0/m_R) B_L(q_0)} \]

\[ (m_{23})^2 = \frac{q_{23}^2}{m_{23}} \]

\[ B_L(q_{23}) \]

\[ B_L(q_0) \]

\[ B_L(q_{23}) \]

no 2-body phase space
Amplitude Model

\[ A_R = p_1 \mu B_L(q_{rM}, d_M) \frac{-g^{\mu\nu} + \frac{p_R^\mu p_R^\nu}{p_R^2}}{s_{23} - m_R^2 - i m_R \Gamma(m_{23})} B_L(q_{rR}, d_R) q_{23} \nu \]

\[ \mathcal{M}_{fi} = \sum_R c_R e^{i \theta_R} A_R(s_{12}, s_{23}) \]

sensitivity to phases is one of the key reasons amplitude analyses are so interesting.

\[ P(s_{12}, s_{23}) = \frac{\left| \mathcal{M}_{fi} \right|^2 \left| \frac{d\Phi}{ds_{12} ds_{23}} \right|}{\int \left| \mathcal{M}_{fi} \right|^2 \left| \frac{d\Phi}{ds_{12} ds_{23}} \right| ds_{12} ds_{23}} \]

\[ = \frac{\left| \mathcal{M}_{fi} \right|^2}{\int \left| \mathcal{M}_{fi} \right|^2 ds_{12} ds_{23}} \]

within kin boundary

Fitters frequently used at LHCb:
- MINT (esp for >3 body)
- AmpGen (descendent of MINT)
- Laura++
- z-fit/tensor flow and
- GooFit-based fitters
Isobar Model + non-resonant

\[ M = \text{graph} + \text{graph} + \text{graph} + \text{graph} + \text{graph} + \ldots \]

\[ = c_0 + c_1 \frac{1}{s_{23} - m_1^2 - i m_1 \Gamma_1(s_{23})} + \ldots \]

- “Classic” experimentalists’ model: Isobar model with resonances described by Breit Wigner lineshapes, plus “non-resonant” term (also called “contact term”, and, amongst theorists, “background”).

- Despised by theorists - especially the non-resonant term (but in practice it’s often needed to describe the data).
Sum of Breit Wigners
Sum of Breit Wigners with non-resonant term
Amplitude Model

\[ \mathcal{M}_{fi} = \sum_{R} c_R e^{i\theta_R} A_R(s_{12}, s_{23}) \]

element:

<table>
<thead>
<tr>
<th>Resonance</th>
<th>(a)</th>
<th>(\delta [^\circ])</th>
<th>Fit fractions [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K^*(892)^\pm)</td>
<td>1.911 \pm 0.012</td>
<td>132.1 \pm 0.7</td>
<td>61.80 \pm 0.31</td>
</tr>
<tr>
<td>(K_0^*(1430)^\pm)</td>
<td>2.093 \pm 0.065</td>
<td>54.2 \pm 1.9</td>
<td>6.25 \pm 0.25</td>
</tr>
<tr>
<td>(K_2^*(1430)^\pm)</td>
<td>0.986 \pm 0.034</td>
<td>308.6 \pm 2.1</td>
<td>1.28 \pm 0.08</td>
</tr>
<tr>
<td>(K^*(1410)^\pm)</td>
<td>1.092 \pm 0.069</td>
<td>155.9 \pm 2.8</td>
<td>1.07 \pm 0.10</td>
</tr>
<tr>
<td>(\rho(770))</td>
<td>1</td>
<td>0</td>
<td>18.85 \pm 0.18</td>
</tr>
<tr>
<td>(\omega(782))</td>
<td>0.038 \pm 0.002</td>
<td>107.9 \pm 2.3</td>
<td>0.46 \pm 0.05</td>
</tr>
<tr>
<td>(f_0(980))</td>
<td>0.476 \pm 0.016</td>
<td>182.8 \pm 1.3</td>
<td>4.91 \pm 0.19</td>
</tr>
<tr>
<td>(f_2(1270))</td>
<td>1.713 \pm 0.048</td>
<td>329.9 \pm 1.6</td>
<td>1.95 \pm 0.10</td>
</tr>
<tr>
<td>(f_0(1370))</td>
<td>0.342 \pm 0.021</td>
<td>109.3 \pm 3.1</td>
<td>0.57 \pm 0.05</td>
</tr>
<tr>
<td>(\rho(1450))</td>
<td>0.709 \pm 0.043</td>
<td>8.7 \pm 2.7</td>
<td>0.41 \pm 0.04</td>
</tr>
<tr>
<td>(f_0(600))</td>
<td>1.134 \pm 0.041</td>
<td>201.0 \pm 2.9</td>
<td>7.02 \pm 0.30</td>
</tr>
<tr>
<td>(\sigma_2)</td>
<td>0.282 \pm 0.023</td>
<td>16.2 \pm 9.0</td>
<td>0.33 \pm 0.04</td>
</tr>
<tr>
<td>(K^*(892)^\pm)(DCS)</td>
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$$M_{fi} = \sum R c_R e^{i\theta_R} A_R(s_{12}, s_{23}) + a_0 e^{i\theta_0}$$

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CDF: PHYSICAL REVIEW D 86, 032007 (2012)

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\[ M_{fi} = \sum_R c_R e^{i \theta_R} A_R(s_{12}, s_{23}) + a_0 e^{i \theta_0} \]

**Example:**

\[ FF_R = \frac{\int \left| c_R e^{i \theta_R} A_R(s_{12}, s_{23}) \right|^2 ds_{12} ds_{23}}{\int \left| \sum_j c_j e^{i \theta_j} A_j(s_{12}, s_{23}) \right|^2 ds_{12} ds_{23}} \]

**Table II:** Combined constrained masses and widths of the included resonances, in the example:

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example:

CDF: PHYSICAL REVIEW D 86, 032007 (2012)
Mixing formalism for 2-body WS decays

\[ \frac{\Gamma(D^0 \rightarrow K^+\pi^-)}{\Gamma(D^0 \rightarrow K^-\pi^+)}(t) \approx (r_{K\pi}^D)^2 + r_{K\pi}^D y_{K\pi}' \Gamma t + \frac{x_{K\pi}'^2 + y_{K\pi}'^2}{4} (\Gamma t)^2 \]

where

\[ \begin{pmatrix} x_{K\pi}' \\ y_{K\pi}' \end{pmatrix} = \begin{pmatrix} \cos \delta_{K\pi} & \sin \delta_{K\pi} \\ \cos \delta_{K\pi} & -\sin \delta_{K\pi} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
Time-dependent CPV $D^0 \rightarrow K_{S\pi\pi}$

$$D^0 \rightarrow K_{S\pi\pi}$$

Fit case  Parameter | Fit new result
--- | ---
No CPV $x(\%)$ | $0.56 \pm 0.19^{+0.03+0.06}_{-0.09-0.09}$
No CPV $y(\%)$ | $0.30 \pm 0.15^{+0.04+0.03}_{-0.05-0.06}$
No dCPV $|q/p|$ | $0.90^{+0.16+0.05+0.06}_{-0.15-0.04-0.05}$
No dCPV arg $q/p(\%)$ | $-6 \pm 11^{+3+3}_{-3-4}$

(Belle preliminary) (by now published)

Time-dependent CPV $D^0 \to K_{S\pi\pi}$

> \[ \frac{q}{p} = \frac{q}{p} e^{i\phi_{CP}} D^0 \]

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Magic of Dalitz plot (sensitivity to phases) gives access to $x$, $y$ (rather than $x''$ and $y''$)

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Magic of Dalitz plot (sensitivity to phases) gives access to $x$, $y$ (rather than $x'^2$ and $y'$)

No evidence of CP violation

Time-dependent CPV $D^0 \rightarrow K_{S\pi\pi}$

\[ \frac{q}{p} = \left| \frac{q}{p} \right| e^{i\phi_{CP}} \cdot D^0 \]

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Magic of Dalitz plot (sensitivity to phases) gives access to $x$, $y$ (rather than $x''$ and $y''$)

No evidence of CP violation

Significant systematic uncertainty from amplitude model dependence. (Could be limiting with future LHCb/upgrade statistics.)

“Isobar” Model

• “Isobar”: Describe decay as series of 2-body processes.

\[
\text{\begin{tikzpicture}
  \node (a) at (0,0) {\mathrm{\textbf{\textless}}};
  \node (b) at (1,0) {+};
  \node (c) at (2,0) {\mathrm{\textbf{\textgreater}}};
  \node (d) at (3,0) {\mathrm{\textbf{\textgreater}}};
  \node (e) at (4,0) {\mathrm{\textbf{\textless}}};
  \node (f) at (5,0) {+};
  \node (g) at (6,0) {\mathrm{\textbf{\textgreater}}};
  \node (h) at (7,0) {+};
  \node (i) at (8,0) {\mathrm{\textbf{\textgreater}}};
  \node (j) at (9,0) {\mathrm{\textbf{\textless}}};
  \node (k) at (10,0) {\cdots};
\end{tikzpicture}}
\]

• Usually: each resonance described by Breit Wigner lineshape (or similar) times factors accounting for spin.

• Popular amongst experimentalists, less so amongst theorists: violates unitarity. But not much as long as resonances are reasonably narrow, don’t overlap too much.

• General consensus: Isobar OK-ish for P, D wave, but problematic for S-wave. Alternatives exist, e.g. K-matrix formalism, which respects unitarity.
Isobar Model with sum of Breit Wigners

\[
\frac{1}{s_{12} - m_1^2 - i m_1 \Gamma_1(s_{12})} + \frac{1}{s_{12} - m_2^2 - i m_2 \Gamma_2(s_{12})} + \frac{1}{s_{12} - m_3^2 - i m_3 \Gamma_3(s_{12})} \ldots
\]

- Single narrow resonance well described by Breit Wigner
- Overlapping broad resonances less so. Theoretically problematic: violates unitarity. From a practical point of view problematic as you might get the wrong phase motion.
Isobar Model with sum of Breit Wigners

\[
\begin{align*}
R_1 & \quad + \quad R_2 & \quad + \quad R_3 & \quad \ldots \\
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\end{align*}
\]
Flatté Formula

- Consider $f_0(980)$ (width $\Gamma \approx 40$-100 MeV). Decays to $\pi \pi$ and $KK$. To $KK$ only above $\sim 987.4$ MeV.

- The availability of the $KK$ final state above 987.4 MeV increases the phase space and thus the width above this threshold.

- Need to take this into account even if I only look at $f_0(980) \rightarrow \pi \pi$.

$$\Gamma_{f_0}(s) = \Gamma_{\pi}(s) + \Gamma_K(s)$$

\[
\Gamma_{\pi}(s) = g_{\pi} \sqrt{s/4 - m_{\pi}^2}, \\
\Gamma_K(s) = \frac{g_K}{2} \left( \sqrt{s/4 - m_{K^+}^2} + \sqrt{s/4 - m_{K^0}^2} \right)
\]
K-matrix

\[ S_{fi} = \langle f | S | i \rangle = I + 2iT \]
\[ T = K(I - iK)^{-1} \]
\[ K_{ij} = \sum_{\alpha} \frac{\sqrt{m_{\alpha} \Gamma_{\alpha i}} \sqrt{m_{\alpha} \Gamma_{\alpha j}}}{m_{\alpha}^2 - m^2} \]

- For single channel: Reproduces Breit Wigner
- For single resonance that can decay to different final state: Reproduces Flatté.
Consider two poles in a single channel:

\[ S_{fi} = \langle f | S | i \rangle = I + 2iT \]

\[ T = K(I - iK)^{-1} \]

\[ K_{ij} = \sum_{\alpha} \frac{\sqrt{m_\alpha \Gamma_{\alpha i}} \sqrt{m_\alpha \Gamma_{\alpha j}}}{m_\alpha^2 - m^2} \]
K-matrix

• Note that the K-matrix approach is still an approximation.

• While it ensures unitarity (by construction), it is not completely theoretically sound/motivated (and violates analyticity).

• And it does not in any way address this:
What theorists think of all this
Sum of Breit Wigners
Sum of Breit Wigners with non-resonant term
Amplitude Models and their issues

- Let’s look at this (effectively 2-body scattering) problem, first:

\[ \text{Diagram of three-body decay processes} \]

Describing this with sums of Breit Wigner line shapes violates some fundamental principles, in particular unitarity (which then breaks the relation between magnitude and phase of the amplitude). OK-ish for narrow resonances that do not overlap too much.

- We’ll postpone the discussion this for a few slides:
Amplitude Models and their issues

• Let’s look at this (effectively 2-body scattering) problem, first:

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\begin{array}{c}
\text{\quad + \quad + \quad \ldots.}
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Three-body decays:

\( V \rightarrow 3\pi \)

Unitarity relation for \( \mathcal{F}(s) \):

\[
\text{disc} \mathcal{F}(s) = 2i \left\{ \mathcal{F}(s) \right\}_{\text{right-hand cut}} + \hat{\mathcal{F}}(s) \text{left-hand cut} \times \theta(s - 4M_\pi^2) \times \sin \delta_1(s) e^{-i\delta_1(s)} \]

\( \hat{\mathcal{F}}(s) \): angular average over the \( t \), \( u \)-channel,

\[
\mathcal{F}(s) = a \Omega(s) \left\{ 1 + \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\sin \delta_1(s') \hat{\mathcal{F}}(s')}{s' |\Omega(s')|(s' - s)} \right\}
\]

Omnès takes into account just this

\( \hat{\mathcal{F}}(s) = \frac{3}{2} \int_{-1}^{1} dz (1 - z^2) \mathcal{F}(t(s, z)) \)

B. Kubis, Three-body decays beyond the isobar model – p. 8

→ crossed-channel scattering between \( s \), \( t \)-, and \( u \)-channel

Bastian Kubis
For each set, the fit fractions are calculated. The distributions of the obtained fit fraction parameter values from the fit are used to generate 500 data-size sample parameter sets.

Figure 12: The 7R model fit gives the ratio of observed decays into \( J/\psi \pi \pi \) to \( 0 \), where the uncertainties are statistical and systematic, respectively; wherever two uncertainties are shown, the lower uncertainty is statistical, and the upper uncertainty is systematic. The points with error bars are data compared with the corresponding phases. In the elastic region, the phase of the signal is visible.

Daub, Hanhart, Kubis: JHEP 1602 (2016) 009
Formalism applied to $\phi \rightarrow \pi\pi\pi^0$

Experimental comparison to $\phi \rightarrow 3\pi$

- successive slices through Dalitz plot:

$$\chi^2/\text{ndof} \quad 1.7 \ldots 2.1$$

$\rightarrow$ pairwise interaction only (with correct $\pi\pi$ scattering phase)
Formalism applied to $\phi \to \pi\pi\pi^0$

Experimental comparison to $\phi \to 3\pi$

- successive slices through Dalitz plot: Niecknig, BK, Schneider 2012

\[ \chi^2/\text{ndof} \quad 1.7 \ldots 2.1 \quad 1.2 \ldots 1.5 \]

→ full 3-particle rescattering, only overall normalization adjustable
Formalism applied to $\phi \rightarrow \pi \pi \pi^0$

**Experimental comparison to $\phi \rightarrow 3\pi$**

- successive slices through Dalitz plot: \cite{Niecknig2012}

\begin{align*}
\chi^2/\text{ndof} & \quad 1.7 \ldots 2.1 \quad 1.2 \ldots 1.5 \quad 1.0 \\
\rightarrow \quad \text{full 3-particle rescattering, 2 adjustable parameters} \\
& \quad \text{(additional "subtraction constant" to suppress inelastic effects)}
\end{align*}
Formalism applied to $\phi \rightarrow \pi\pi\pi^0$

**Experimental comparison to $\phi \rightarrow 3\pi$**

- successive slices through Dalitz plot: Niecknig, BK, Schneider 2012

![Dalitz plot](image)

$\chi^2/\text{ndof}$

|   | 1.7 ... 2.1 | 1.2 ... 1.5 | 1.0 |

- perfect fit respecting analyticity and unitarity possible
- contact term emulates neglected rescattering effects
- no need for "background" — inseparable from "resonance"
Formalism applied to $D \rightarrow \pi\pi\pi K$

Fit limited to $M(K\pi) < M(\eta') + M(K) \approx 1.45\text{GeV}$
elastic approximation breaks down beyond.

- Omnès fit: $\chi^2/\text{ndof} \approx 1.42$
  ("isobar model" + non-resonant background waves)
- full dispersive solution: $\chi^2/\text{ndof} \approx 1.11$
  $\rightarrow$ visible improvement similar to $\phi \rightarrow 3\pi$
- full fit in terms of 7 complex subtraction constants
  (−1 phase, −1 overall normalisation)

Niecknig, BK in progress
Three-body FSI (beyond 2+1)

\[
\begin{align*}
M_{\text{FSI}} = & \quad \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \cdots \\
\end{align*}
\]

- shown to be relevant on charm sector

\[
D^+ \rightarrow K^- \pi^+ \pi^+
\]
• Three-body FSI (beyond 2+1)

\[ M \text{FSI} = M_{2\text{-body}} + M_{3\text{-body}} + \ldots \]

• Shown to be relevant on charm sector

\[ D^+ \to K^- \pi^+ \pi^+ \]

\[(2\text{-body phase})\]

Models available

PRD92 094005 (2015)
Models available

- Three-body FSI (beyond 2+1)

\[ M = \text{FSI} \]

- Shown to be relevant on charm sector

\( D^+ \to K^- \pi^+ \pi^+ \)

3-body approaches

- Niecknig, Kubis, JHEP10 142 (2015)

- 3-body FSI play a role
- Data analysis…
multi meson model - $D^+ \rightarrow K^- K^+ K^+$

- alternative to isobar model
- starts from Chiral Theory

\[ A_{ab}^{JI} \]  
unitary scattering amplitude for $ab \rightarrow K^+ K^-$

full FSI: coupled channel
multi meson model - $D^+ \rightarrow K^- K^+ K^+$

- alternative to isobar model
- starts from Chiral Theory
  - $A_{ab}^{JI}$ → unitary scattering amplitude for $ab \rightarrow K^+ K^-$
  - full FSI: coupled channel
  - parameters have physical meaning: resonance masses and coupling constants
  - relative phase between partial waves is NOT fitted
Theoretical model

Multimeson model for the $D^* \to K^* K^- K^+$ decay amplitude

R. T. Aoude, P. C. Magalhães, A. C. dos Reis, and M. R. Robilotta

\[ T^S = T^S_{NR} + T^{00} + T^{01} \]

\[ T^P = T^P_{NR} + T^{11} + T^{10} \]

- free parameters

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$94.3^{+2.8}_{-1.7} \pm 1.5$ MeV</td>
</tr>
<tr>
<td>$m_{a_0}$</td>
<td>$947.7^{+5.5}_{-5.0}$ MeV</td>
</tr>
<tr>
<td>$m_{S_0}$</td>
<td>$992.0^{+8.5}_{-7.5}$ MeV</td>
</tr>
<tr>
<td>$m_{S_1}$</td>
<td>$1330.2^{+5.9}_{-6.5}$ MeV</td>
</tr>
<tr>
<td>$m_\phi$</td>
<td>$1019.54^{+0.10}_{-0.10} \pm 0.51$ MeV</td>
</tr>
<tr>
<td>$G_\phi$</td>
<td>$0.464^{+0.013}_{-0.009} \pm 0.007$</td>
</tr>
<tr>
<td>$c_d$</td>
<td>$-78.9^{+4.2}_{-2.7} \pm 1.9$ MeV</td>
</tr>
<tr>
<td>$c_m$</td>
<td>$106.0^{+7.7}_{-4.6} \pm 3.3$ MeV</td>
</tr>
<tr>
<td>$\tilde{c}_d$</td>
<td>$-6.15^{+0.55}_{-0.54} \pm 0.19$ MeV</td>
</tr>
<tr>
<td>$\tilde{c}_m$</td>
<td>$-10.8^{+2.0}_{-1.5} \pm 0.4$ MeV</td>
</tr>
</tbody>
</table>

![Graphs and tables showing the fitted data](image)

→ good fit with fewer parameters than the isobar

Hadronic FSI - The University of Manchester - 13th November 2020 - Patricia Magalhães
Any 3-body decay amplitude

\[ T = T + W_A + W_A + W_F + W_F + W_K + K_K + K_K + \ldots \]

(2+1) approach

Form factor

meson-meson

\[ T = W + W_A + W_F \]

\[ F = \__ + A \]

\[ A = K + K_K + K_K + \ldots \]
Tool kit for meson-meson interactions in 3-body decay

- Any 3-body decay amplitude

\[ T = T + W A + W A + W F + \ldots \]

(2+1) approach

\[ T = W + W A + W F + \ldots \]

Form factor

\[ F = \ldots + A \]

meson-meson

\[ A = K + K K + K K K + \ldots \]

provide the building block \( A \)

- includes multiple resonances in the same channel (as many as wanted)
- free parameter (masses and couplings) to be fitted to data.

Available to be implement in data analysis!!
Three $\pi^-\pi^+$ S-wave parametrisation in $B^-\rightarrow\pi^-\pi^+\pi^-$

S-wave model projections

| $|A|^2$ | Phase |
|--------|--------|
| ![Isobar](image1) | ![Phase](image2) |
| ![K-matrix](image3) | ![QMI](image4) |

CP violation is pretty evident here!

slide by Daniel O’Hanlon
Model-Independent line shapes

a1(1260) line shape in $D \rightarrow \pi\pi\pi\pi$, CLEO-data
Model-Independent line shapes

Use known resonances as interferometer to obtain model-independent amplitude and phase of resonance.
Example: LHCb Pentaquark discovery in $\Lambda_b^0 \rightarrow J/\psi p K^-$
Fully model-independent

Use as input only the well-understood angular distributions of S, P, D, F, … resonances (expanded in terms of Legendre Polynomials) in $K^-p$, and the measured $K^-p$ mass spectrum, to model $J/\psi p$ mass spectrum (blue). Is the peak just a reflection? No!

This means it’s not the kind of reflection introduced earlier with the example of $D \rightarrow K_S \pi \pi$.
See also

  Very useful especially when dealing with complicated spin structures.

  Identifies and overcomes difficulties in model-independent amplitude analyses.
Das Model

The Model

- There are, for most cases we care about, no theoretically sound amplitude models...

- However, there are “good enough” models. What’s good enough depends on the purpose.

- So what to do? Suggest a mix of....
  - model-independent approaches
  - “good enough” models of various levels of sophistication
  - improve models (there is - and that’s fairly new - real, tangible, progress!)