Belle II physics week: Amplitude analysis theory

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For a practical guide on how to treat resonances:
See review "Resonances" in the Review of particle physics by the PDG (under "kinematics")

In what follows I heavily borrowed from J.R. Peláez & B. Kubis

analyticity $\Longleftrightarrow$ causality
unitarity $\Longleftrightarrow$ probability conservation

Connection of analyt. & causality

this will be illustrated using the classical damped harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \qquad (1)$$

$\gamma > 0$ (damping)

a) free solution ($f(t) = 0$)

we chose the ansatz $x_f(t) = a e^{i\omega t}$ (2)

inserting (2) into (1):

$$\omega^2 - 2i\gamma\omega - \omega_0^2 = 0$$

$$\leadsto \omega_{1/2} = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2} = i\gamma \pm \omega_\gamma$$

we want to focus on weak damping: $\gamma \ll \omega_0$

$\leadsto \omega_\gamma$ is real

$$x_f(t) = \left(a_+ e^{i\omega_\gamma t} + a_- e^{-i\omega_\gamma t}\right) e^{-\gamma t}$$

damping, since $\gamma > 0$

b) harmonic force: $f(t) = F_\omega e^{-i\omega t}$

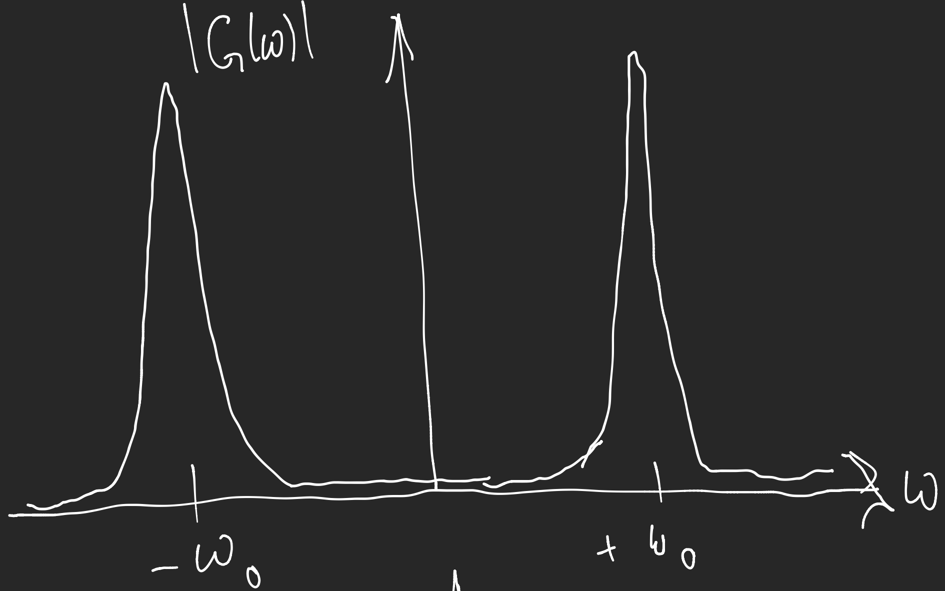
now we use the ansatz: $x(t) = F_\omega G(\omega) e^{-i\omega t}$

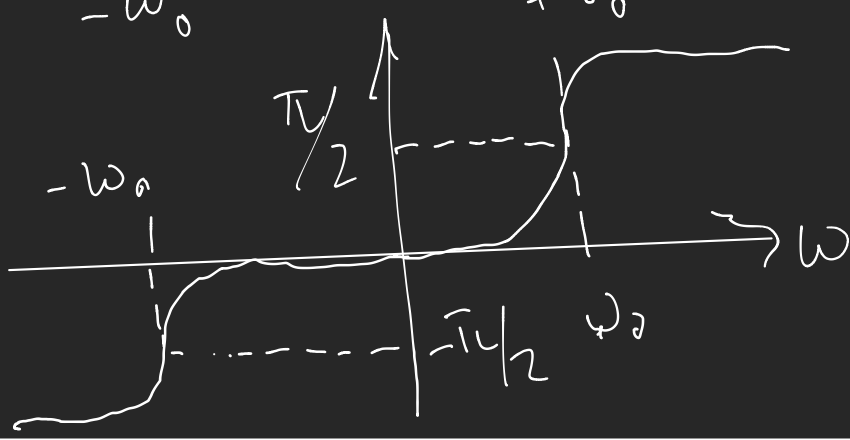
& we get from putting this into the Eq. of motion (1)

$$G(\omega)\underbrace{\left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right)}_{= -(\omega-\omega_1)(\omega-\omega_2)} = 1$$

$$\boxed{G(\omega) = \frac{-1}{(\omega-\omega_1)(\omega-\omega_2)}}$$

for weak damping:





for external excitations with $\omega \gtrsim \omega_{1/2}$ we find resonances characterised by a optimal energy of the external force to cut system

For an arbitrary external excitation one can always write:
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; F(\omega)\, e^{-i\omega t}$$

& analogously for the solution
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, G(\omega) F(\omega)\, e^{-i\omega t} + x_f(t)$$

since $F(\omega) = \int_{-\infty}^{\infty} dt'\, f(t')\, e^{i\omega t'}$ we get

$$x(t) = \int_{-\infty}^{\infty} dt'\, f(t')\, g(t-t')$$

where $g(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, G(\omega)\, e^{-i\omega(t-t')}$

$g(\tau)$ corresponds to the propagation of an instantaneous force $f(t') = \alpha\,\delta(t')$ from t' to $t'+\tau$

$$\boxed{\text{Causality implies that } g(\tau<0) = 0}$$

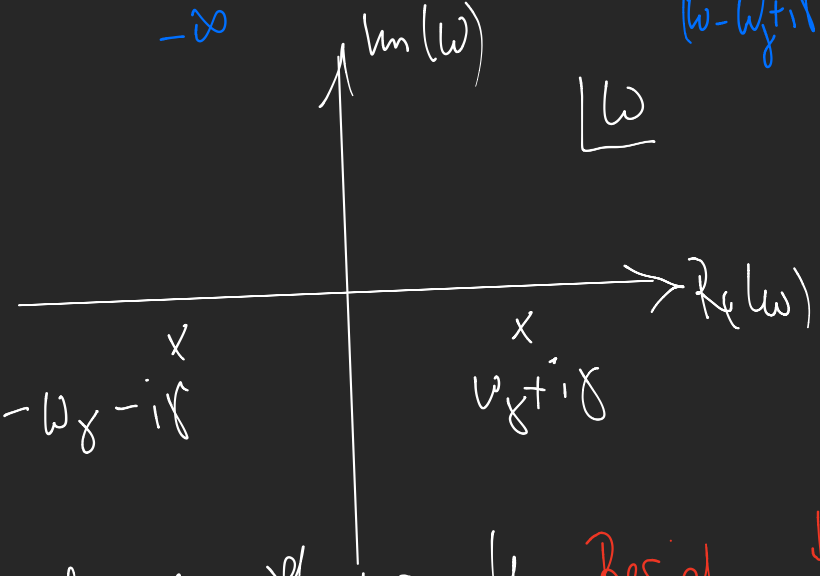
for otherwise a force that will operate in the future would impact the present.

To see that $g(\tau)$ is indeed causal, we need to look at its analytic structure

(this is the connection we were looking for)

we had
$$g(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, G(\omega) e^{-i\omega\tau} \qquad G(\omega) = \frac{-1}{(\omega-\omega_\gamma+i\gamma)(\omega+\omega_\gamma+i\gamma)}$$

with $\gamma > 0$



To evaluate the integral we will use the Residue theorem

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} I(\gamma, a_k)\, \mathrm{Res}(f, a_k)$$

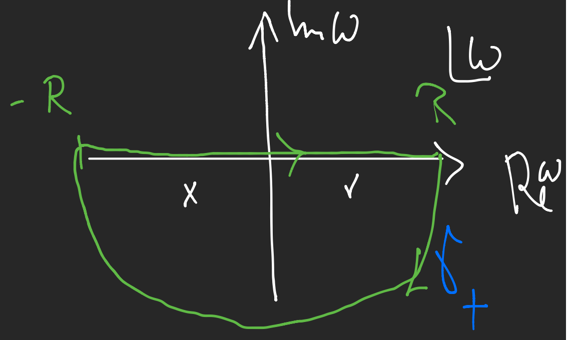
if γ is a positively oriented simple closed curve

then
$$\underset{\sim}{I}(\gamma, a_k) = \begin{cases} 1 & \text{if } a_k \text{ is enclosed by } \gamma \\ 0 & \text{otherwise} \end{cases}$$

so to apply this to our $\int_{-\infty}^{\infty} d\omega \ldots$ we need to close the contour in a way that the add. part of the path does not contribute to the integral

The relevant factor for this is $\boxed{e^{-i\omega\tau}}$

- for $\tau > 0$ we need to close the contour in the lower half plane, since then on the arc we have $\mathrm{Im}(\omega) < 0 \rightsquigarrow$



$$-i(i\,\mathrm{Im}(\omega))\tau = \mathrm{Im}(\omega)\tau < 0$$

& we may write:

$$g(\tau)\Big|_{\tau>0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, G(\omega) e^{-i\omega\tau}$$

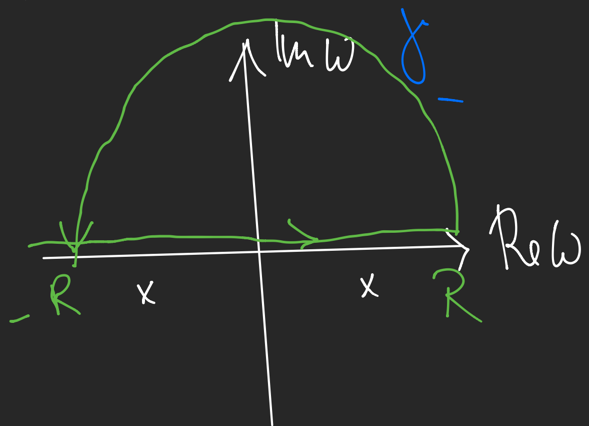
$$= \lim_{R\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, G(\omega) e^{-i\omega\tau}$$

$$= \frac{1}{2\pi}(2\pi i)\left(-\frac{-1}{\omega_1 - \omega_2} e^{-i\omega_1\tau} - \frac{-1}{\omega_2-\omega_1} e^{-i\omega_2\tau}\right)$$

$$\overset{\text{math. neg.}}{=} i\frac{1}{2\omega_\gamma}\left(e^{-i\omega_\gamma\tau} - e^{+i\omega_\gamma\tau}\right)e^{-\gamma\tau} = \frac{1}{\omega_\gamma}\sin(\omega_\gamma\tau)e^{-\gamma\tau}$$

- for $\tau < 0$ we now need to close the contour in the upper half plane

$$g(\tau)\Big|_{\tau<0} = \lim_{R\to\infty}\frac{1}{2\pi}\int_{\gamma_-} d\omega\, G(\omega) e^{-i\omega\tau}$$



$= 0$ as required from causality

thus we found:

$$\boxed{G(\omega) \text{ analytic in the upper half plane} \iff \text{causality } g(\tau<0)=0}$$

one can show that non-rel. scattering amplitudes are causal in this spirit & as such have to have a well defined analytic structure. The is adopted for relat. amplitudes.

Implications of unitarity & analyt. for physical scattering amplitudes

lets look at some scattering process

In quantum mechanics the time evolution of some system is provided by the time evolution operator

$$\hat{U}(t',t)\,|\Psi(t)\rangle = |\Psi(t')\rangle \quad \text{with} \quad \hat{U}(t',t) = e^{-i\hat{H}(t'-t)}$$

thus we may introduce the S-matrix via

$$\hat{S} = \lim_{t' \to +\infty} \lim_{t \to -\infty} \hat{U}(t',t)$$

Properties of S:

- by definition we have $\hat{U}(t,t') = \hat{U}(t',t)^{-1}$ or equivalently: $\hat{U}(t,t')\,\hat{U}(t',t) = \mathbb{1}$
  moreover: $\hat{U}(t',t) = \hat{U}(t,t')^{+}$ since $\hat{H} = \hat{H}^{+}$

thus $\mathcal{U}$ is unitary & thus : $\boxed{S^{\dagger}S = \mathbb{1}}$

This is nothing but the conservation of probability.

- In the absence of interactions : $\hat{S} = \mathbb{1}$
- Causality demands that S is analytic in the whole complex plane of the physical sheet besides the real axis

Thus non-trivial scattering comes from $S - \mathbb{1}$ ⇝
For 2-body scattering we define

$$ {}_{out}\langle p_1', p_2', b | \hat{S} - \mathbb{1} | p_1, p_2, a \rangle_{in} = i (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \times \mathcal{M}(p_1, p_2; p_1' p_2')_{ba} $$

Where single particle states are normalised accor. to

$$ \langle p' | p \rangle = (2\pi)^3 2E_p \, \delta^{(3)}(\vec{p} - \vec{p}') $$

From the unitarity of the S-matrix we get :

$$ \underline{\hat{\mathcal{M}}_{ba} - \hat{\mathcal{M}}^*_{ab}} = \sum_c \hat{\mathcal{M}}^*_{cb} \hat{\mathcal{M}}_{ca} = 2i \, \mathrm{Im}(\hat{\mathcal{M}}) $$

Where for the last equality I use $\mathcal{M}(s^*) = \mathcal{M}(s)^*$

Introducing basis states we get

$$\mathrm{Disc}(\mathcal{M}_{ba}) = i(2\pi)^4 \sum_c \int d\Phi_c \, \mathcal{M}^*_{cb} \mathcal{M}_{ca}$$

the phase space is here defined via

$$d\Phi_n(P; p_1, \ldots, p_n) = \delta^{(4)}\left(P - \sum_{i=1}^n p_i\right) \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

For only 2-particle channels this gives

$$\mathrm{Im}\, \mathcal{M}_{ba} = \sum_c \mathcal{M}^*_{cb} \rho_c \mathcal{M}_{ca} \, ; \quad \rho_c = \frac{1}{8\pi} \frac{|\vec{q}_c|}{\sqrt{s}} \Theta(s - s_{thr})$$

for prod. amplitudes one finds

$$\mathrm{Im}\, A_a = \sum_b \mathcal{M}^*_{ba} \rho_b A_b \qquad (*)$$

therefore the scattering Matrix $\mathcal{M}$ develops an imaginary part for $s > s_{thr.}$ & is real below the lowest threshold for real values of s.

This implies: There is a branchpoint @ every $s = s_{thr}$. This branchpoint is of $\sqrt{\ }$-type, since $q_c \underset{\text{non-rel.}}{=} \sqrt{2\mu E}$ with $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ & $E = 0$ at threshold

Whenever a new channel opens the number of Riemann-Sheets doubles

For 1 channel (elastic): 2 sheets

1st sheet: Physical sheet; contains poles at the real axis below the threshold & a branch point at threshold

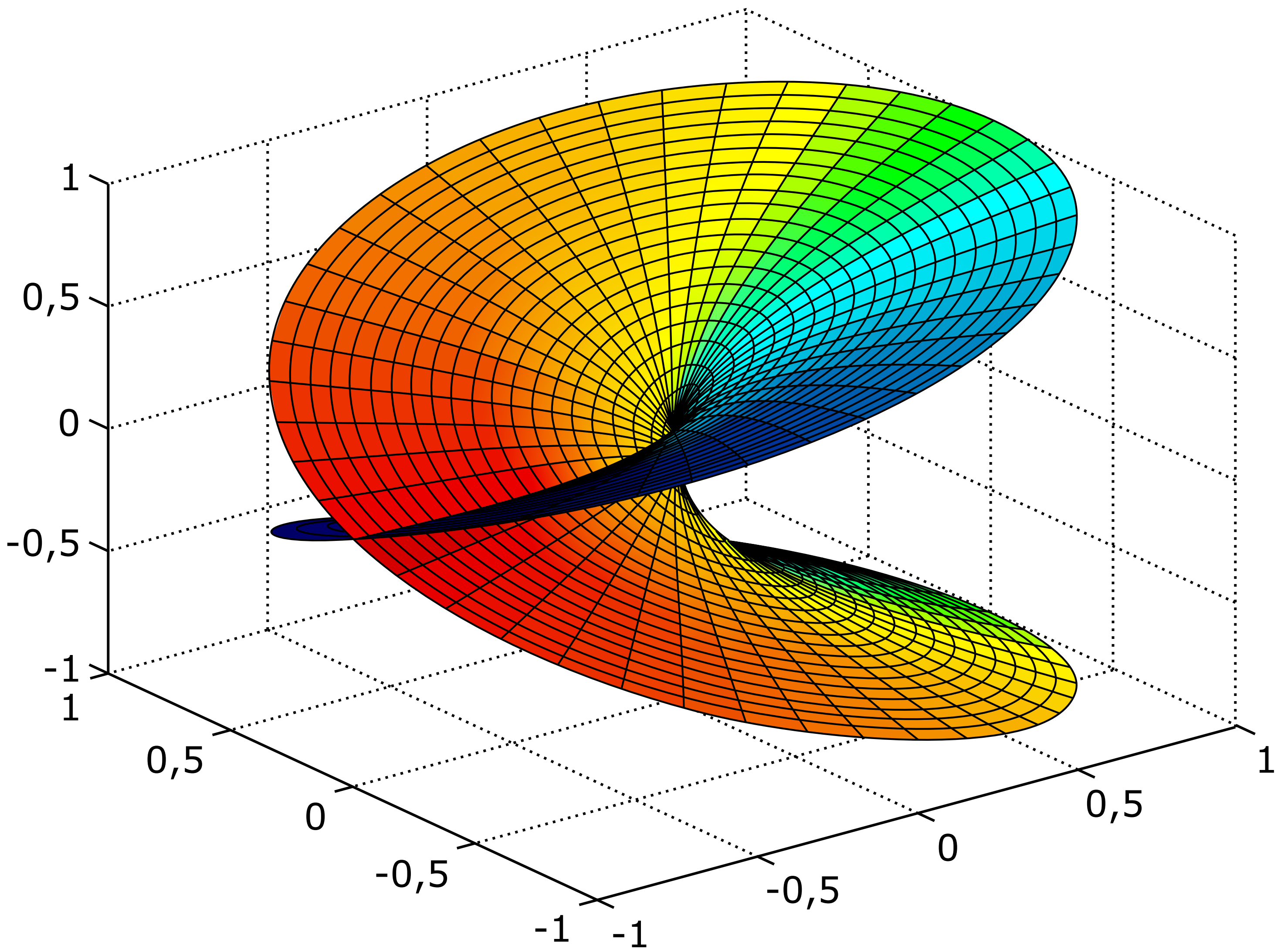
bound states

2nd sheet: unphysical sheet, contains poles either on the real axis below the the threshold (virtual states) or poles in the complex plane, but then they must come in pairs, since $\mathcal{M}(s^*) = \mathcal{M}(s)^*$ (resonances)

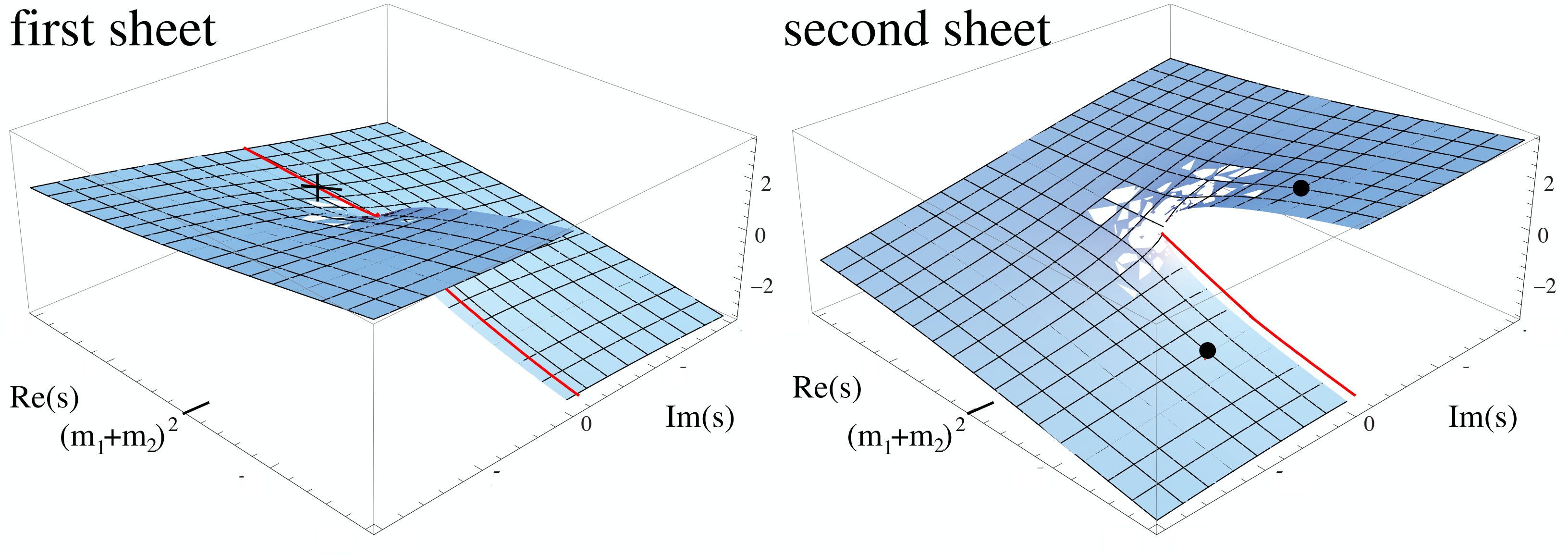
One closing remark on the implication of unitarity for production amplitudes in the 1-channel case: Now relation (*) reads:

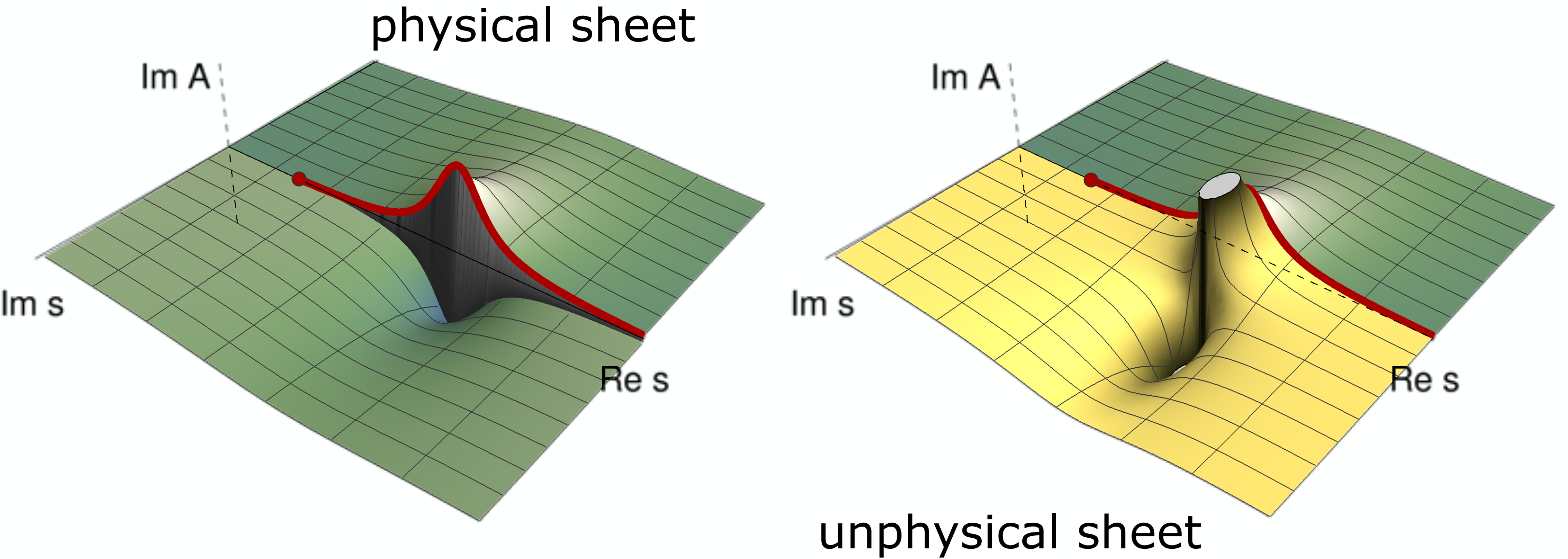
$$\mathrm{Im}\, A = \mathcal{M}^* \rho A$$

& since $\mathrm{Im}\, A \in \mathbb{R}$ ⇝ phase of A must agree to the phase of $\mathcal{M}$ (Watson theorem)



https://upload.wikimedia.org/wikipedia/commons/9/9c/Riemann_sqrt.svg


first sheet
second sheet
Re(s)
$(m_1+m_2)^2$
Im(s)
0
2
0
−2
Re(s)
$(m_1+m_2)^2$
Im(s)
0
2
0
−2


physical sheet
Im A
Im s
Re s
Im A
Im s
Re s
unphysical sheet