For a practical guide on how to treat resonances:

See review "Resonances" in the Review of Particle physics by the PDG (under "kinematics")

In what follows I heavily borrowed from J.R. Peláez & B. Kubis

\[ x'' + 2\gamma x' + \omega_0^2 x = f(t), \quad \gamma > 0 \] (damping)

\[ x(t) = a e^{i\omega t} \] \hspace{1cm} (2)

a) free solution \((f(t)=0)\): we chose the ansatz \(x(t) = a e^{i\omega t} \)

\[ \omega^2 = 2\gamma \omega - \omega_0^2 = 0 \]

\[ \omega = \omega_0 \pm i\gamma \left( \pm \sqrt{\omega_0^2 - 2\gamma^2} \right) = \omega_0 \pm i\gamma \]

we went to focus on weak damping: \(\gamma \ll \omega_0\)
no \( \omega_f \) is real
\[ X_f(t) = (A^+ e^{i \omega_f t} + A^- e^{-i \omega_f t}) e^{-\zeta t} \]
damping, since \( \zeta > 0 \)

b) harmonic force: \( f(t) = \bar{F} e^{-i \omega t} \)
now we use the ansatz: \( x(t) = \bar{F} \omega G(\omega) e^{-i \omega t} \)
& we get from putting this into the Eq. of motion (1)
\[ G(\omega) \left( -\omega^2 + 2i \zeta \omega + \omega_0^2 \right) = 1 \]
\[ = -(\omega - \omega_1)(\omega - \omega_2) \]

for weak damping:

\[ |G(\omega)| \]

\[ -\omega_0 \quad +\omega_0 \]

\[ -\omega \quad +\omega \]

\[ -\pi/2 \quad +\pi/2 \]

\[ \omega \]

\[ \bar{F} \omega \]

\[ G(\omega) = \frac{-1}{(\omega - \omega_1)(\omega - \omega_2)} \]
for external excitations with $\omega \approx \omega_n$ we find resonances characterised by an optimal energy of the external force to our system.

For an arbitrary external excitation one can always write:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ F(\omega) e^{-i\omega t}$$

And analogously for the solution:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ G(\omega) F(\omega) e^{-i\omega t} + x_f(t)$$

Since $F(\omega) = \int_{-\infty}^{\infty} dt \ f(t') e^{i\omega t'}$ we get:

$$x(t) = \int_{-\infty}^{\infty} dt' \ f(t') g(t-t')$$

where $g(t-t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ G(\omega) e^{-i\omega (t-t')}$

$g(t)$ corresponds to the propagation of an instantaneous force $f(t') = \delta(t')$ from $t'$ to $t+t'$. Causality implies that $g(t<0) = 0$.

For otherwise a force that will operate in the future.
would impact the present.

To see that \( g(t) \) is indeed causal, we need to look at its analytic structure (this is the connection we were looking for).

We had

\[
g(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, G(\omega)e^{-i\omega t}
\]

\[
G(\omega) = \frac{1}{(\omega - i\delta)(\omega + i\delta + i\gamma)}
\]

with \( \delta > 0 \)

To evaluate the integral we will use the Residue Theorem

\[
\oint \frac{f(z)}{z^k} \, dz = 2\pi i \sum_{k=1}^{n} \text{Res} \left( \frac{f(z)}{z^k} \right)
\]

if \( \gamma \) is a positively oriented simple closed curve then

\[
\int_{C} f(z) \, dz = 0 \quad \text{if} \ a_{k} \ \text{is enclosed by} \ f
\]

otherwise

As to apply this to our \( \int_{-\infty}^{\infty} \) we need to close the contour in a way that the added part of the path does not contribute to the integral

The relevant factor for this is

\[
e^{-i\omega t}
\]
\[ \text{for } T > 0 \text{ we need to close the contour in the lower half plane, since then on the arc we have } \text{Im}(\omega) > 0 \Rightarrow -i \left( i \text{Im}(\omega) T = \text{Im}(\omega) T < 0 \right) \]

\& we may write:

\[ g(z) \big|_{T > 0} = \frac{1}{2\pi i} \int_{C} d\omega G_\omega(z) e^{-i\omega T} \]

\[ = \lim_{R \to 0} \frac{1}{2\pi} \int d\omega G_\omega(z) e^{-i\omega z} \]

\[ = \frac{1}{2\pi} \int d\omega \left( \frac{1}{\nu + i\omega} e^{-i\omega \frac{\omega}{\nu}} - \frac{1}{\omega - i\omega \frac{\nu}{\omega}} e^{-i\omega \frac{\omega}{\nu}} \right) \]

\[ \text{where, we get:} \]

\[ = \frac{1}{2\pi} \int d\omega \left( e^{-i\omega z} - e^{i\omega z} \right) e^{-i\omega T} = \frac{1}{\nu} g(z) e^{-i\omega T} \]

\[ \text{for } T < 0 \text{ we now need to close the contour in the upper half plane:} \]

\[ g(z) \big|_{T < 0} = \lim_{R \to 0} \frac{1}{2\pi i} \int_{C} d\omega G_\omega(z) e^{-i\omega T} \]

\[ = 0 \text{ as required from causality.} \]
one can show that non-tel. scattering amplitudes are causal in this spirit & as such have to have a well-defined analytic structure. The is adopted for rel. amplitudes.

Implications of unitarity & analytic for physical scattering amplitudes

let's look at same scattering process

In quantum mechanics the time evolution of some system is provided by the time-evolution operator

$$\hat{U}(t', t) |t\rangle = |t'\rangle$$

with $$\hat{U}(t', t) = e^{-i\hat{H}(t'-t)}$$

thus we may introduce the S-matrix via

$$\hat{S} = \lim_{t' \to +\infty} \lim_{t \to -\infty} \hat{U}(t', t)$$

Properties of $S$:

- by definition we have $$\hat{U}(t', t) = \hat{U}(t', t)^{\dagger}$$ or equivalently: $$\hat{U}(t', t) \hat{U}(t', t)^{\dagger} = 1$$
- moreover: $$\hat{U}(t', t) = \hat{U}(t', t)^{\dagger}$$ since $$\hat{U} = \hat{U}^\dagger$$
Thus $U$ is unitary & thus:

\[ S^*S = \mathbb{1} \]

This is nothing but the conservation of probability.

In the absence of interactions: $\hat{S} = \mathbb{1}$

Causality demands that $S$ is analytic in the whole complex plane of the physical sheet besides the real axis.

Thus non-trivial scattering comes from $S-\mathbb{1} \Rightarrow$ far 2-body scattering we define

\[
\langle p'_1 p'_2 | S - \mathbb{1} | p_1 p_2, a \rangle = \frac{4}{\pi} (2\pi)^3 \delta^3(p_1 + p_2 - p'_1 - p'_2) \times M(p_1, p_2; p'_1, p'_2) b_a
\]

When single particle states are normalised accor. to

\[
\langle p | 1p \rangle = (2\pi)^3 2E \rho \delta^3(p - p')
\]

From the unitarity of the $S$-matrix we get:

\[
\hat{M}_{ab} - \hat{M}_{ba} = \sum_c \hat{\mathcal{M}}^{*}_{cb} \hat{M}_{ca} = 2i \text{Im}(\hat{M})
\]

Where for the last equality I use $M(S^*) = M(S)^*$
Introducing basis states we get

\[ \text{Disc } (M_{\beta c}) = i (2\pi)^4 \sum_c \int d^4 \Phi \, M^+_{\beta b} M_{c c} \]

the phase space is here defined via

\[ d\rho \, (\vec{p}_1, \ldots, \vec{p}_n) = S \, (\vec{p}_1 \ldots \vec{p}_n) \prod_{i=1}^{n} \frac{1}{2\pi} \frac{d^3 p_i}{(2\pi)^3 2E_i} \]

For only 2-particle channels this gives

\[ \text{Im } M_{\beta c} = \sum_b \frac{c}{b} M^+_{\beta b} \delta_{\beta b} \]

Far prod. amplitudes one finds

\[ \text{Im } A_{\alpha} = \sum_{b \loom \loom c} M^+_{\beta b} \delta_{\beta b} A_{\beta} \]  

\(*\)

therefore the scattering matrix \( M \) develops an infinity part for \( s > s_{\text{thr}} \) & is real below the lowest threshold for real values of \( s \).

This implies : There is a branchpoint a every \( s = s_{\text{thr}} \). This branchpoint is of \( \sqrt{\mu} \)-type, since \( \sqrt{2\mu E} \) & \( \text{Im } A_{\alpha} \) with \( \mu = \frac{m_1 m_2}{m_1 + m_2} \) & \( E = 0 \) at threshold.

Whenever a new channel opens the number of Riemann sheets doubles.
For 1 channel (elastic): 2 sheets

1st sheet: Physical sheet; contains poles at the real axis below the threshold & a branch point at threshold

2nd sheet: Unphysical sheet, contains poles either on the real axis below the threshold (virtual states) or poles in the complex plane, but then they must come in pairs, since $M(s^*) = M(s)^*$ (resonances)

One closing remark on the implication of unitarity for production amplitudes in the 1-channel case: Now relation (*) reads:

$$\text{Im } A = M^* \text{ Re } A$$

& since $\text{Im } A \in \mathbb{R}$ no phase of $A$ must agree to the phase of $M$ (Watson theorem)
first sheet

second sheet

\[ \text{Re}(s) = (m_1 + m_2)^2 \]

\[ \text{Im}(s) \]