How to build a model

BELLE II Physics Week, 3 December 2020

Jonas Rademacker

I received much useful input and advice on this talk. Special thanks go to: Philippe d’Argent, Tim Evans, Tim Gershon, Daniel Johnson, Patricia Magalhães, Mike Williams, Mark Whitehead.
Model selection

• ... first we do a detour to 4-body analyses because:

  • they because of the added complexity compared to 3-body analyses, model selection is more complex, so these analysis tend to use a particularly systematic approach

  • I had the feeling that my answer to the question “and what are the main challenges in four-body analyses” on Monday was a bit too short to be useful, so I can rectify that problem at the same time.
4-body amplitude analyses

• 5 parameters instead of 2. (Don’t just use invariant mass-squared parameters: \(s_{12}, s_{23}, s_{34}, s_{123}, s_{234}\)).

• P-odd moments possible

• Phase space density is not flat in canonical parameters.

• Amplitude structure more complicated
Parity in 3-body decays
Parity in 3-body decays
Rotation in 3-body decays
Rotation in 3-body decays
Rotation in 3-body decays
Rotation in 3-body decays
Rotation in 3-body decays
3-body: $P \cong rotation$

No parity violating effects, no $P$-odd moments.
4-body: $P \neq$ rotation
$P(\text{and } T)$-odd moments in $D^0 \rightarrow K^+ K^- \pi^+ \pi^-$

$$C_T \equiv \vec{p}_{K^+} \cdot (\vec{p}_{\pi^+} \times \vec{p}_{\pi^-})$$

odd under reversal $P$

$$C_T > 0$$

$s_{ij} = (p_i + p_j)^2$, $s_{ijk} = (p_i + p_j + p_k)^2$

are $P$-even and cannot describe $P$-odd kinematics possible in 4-body decays.

BaBar: PRD 81 (2010) 111103
BaBar PRD 84, 031103 (R) (2011)

FOCUS: PLB 622 (2005) 239-248
LHCb: JHEP 1410 (2014) 005
P(and T)-odd moments in $D^0 \rightarrow K^+ K^- \pi^+ \pi^-$

$$C_T \equiv \vec{p}_{K^+} \cdot (\vec{p}_{\pi^+} \times \vec{p}_{\pi^-})$$

odd under reversal P

$s_{ij} = (p_i + p_j)^2$, $s_{ijk} = (p_i + p_j + p_k)^2$ are P-even and cannot describe P-odd kinematics possible in 4-body decays.
P(\text{and } T)-\text{odd moments in } D^0 \to K^+ K^- \pi^+ \pi^-

\begin{align*}
C_T &\equiv \overrightarrow{p}_{K^+} \cdot (\overrightarrow{p}_{\pi^+} \times \overrightarrow{p}_{\pi^-}) \\
A_T &\equiv \frac{\Gamma(C_T > 0) - \Gamma(C_T < 0)}{\Gamma(C_T > 0) + \Gamma(C_T < 0)}
\end{align*}

odd under reversal P

\begin{align*}
s_{ij} &= (p_i + p_j)^2, \\
s_{ijk} &= (p_i + p_j + p_k)^2
\end{align*}

are P-even and cannot describe P-odd kinematics possible in 4-body decays.

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\[ C_T \equiv \vec{p}_{K^+} \cdot (\vec{p}_{\pi^+} \times \vec{p}_{\pi^-}) \]

odd under reversal P

\[ A_T \equiv \frac{\Gamma(C_T > 0) - \Gamma(C_T < 0)}{\Gamma(C_T > 0) + \Gamma(C_T < 0)} \]

\[ \overline{A}_T \equiv \frac{\Gamma(-C_T > 0) - \Gamma(-C_T < 0)}{\Gamma(-C_T > 0) + \Gamma(-C_T < 0)} \]

\[ s_{ij} = (p_i + p_j)^2, \quad s_{ijk} = (p_i + p_j + p_k)^2 \]

are P-even and cannot describe P-odd kinematics possible in 4-body decays.

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\[ A_T \equiv \frac{\Gamma(C_T > 0) - \Gamma(C_T < 0)}{\Gamma(C_T > 0) + \Gamma(C_T < 0)} \]

\[ \bar{A}_T \equiv \frac{\Gamma(-C_T > 0) - \Gamma(-C_T < 0)}{\Gamma(-C_T > 0) + \Gamma(-C_T < 0)} \]

\[ A_T = \frac{1}{2} (A_T - \bar{A}_T) \]

Measures CP violation

$s_{ij} = (p_i + p_j)^2$, $s_{ijk} = (p_i + p_j + p_k)^2$

are P-even and cannot describe P-odd kinematics possible in 4-body decays.

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BaBar PRD 84, 031103 (R) (2011)
FOCUS: PLB 622 (2005) 239-248
LHCb: JHEP 1410 (2014) 005
Spin factors 3-body

\[ M = F_D f(p_D + p_C) \sum_{\text{all polarisations}} \varepsilon^* \varepsilon \frac{m_r^2 - m_{AB}^2 - im_r \Gamma_{AB}}{f(p_B - p_A)F_R} \]
Spin factors 3-body

\[ M = F_D f(p_D + p_C) \sum_{\text{all polarisations}} \frac{\varepsilon^* \varepsilon}{m_r^2 - m_{AB}^2 - i m_r \Gamma_{AB}} f(p_B - p_A) F_R \]
Spin factors 3-body

\[ \mathcal{M} = F_D f(p_D + p_C) \left( \sum_{all \ polarisations} \frac{\varepsilon^* \varepsilon}{m_r^2 - m_{AB}^2 - im_r \Gamma_{AB}} \right) f(p_B - p_A) F_R \]

\begin{align*}
\text{spin 0:} & \quad \frac{1}{(\text{GeV})^2} \\
\text{spin 1:} & \quad T_{\mu \alpha} \quad (p_B^\alpha - p_A^\alpha) \quad (p_B^\beta - p_A^\beta) \\
\text{spin 2:} & \quad T_{\mu \nu \alpha \beta} \quad (p_B^\alpha - p_A^\alpha) (p_B^\beta - p_A^\beta) \\
\end{align*}
Spin factors 3-body

\[
\mathcal{M} = F_D f(p_D + p_C) \sum_{\text{all polarisations}} \frac{\varepsilon^* \varepsilon}{m_r^2 - m_{AB}^2 - im_r \Gamma_{AB}} f(p_B - p_A) F_R
\]

spin 0:
spin 1:
spin 2:

\[
T_{\mu\alpha} = -g_{\mu\alpha} + \frac{p_\mu p_\alpha}{m_r^2}
\]

\[
T_{\mu\nu\alpha\beta} = \frac{1}{2} (T_{\mu\alpha} T_{\nu\beta} + T_{\mu\beta} T_{\nu\alpha}) - \frac{1}{3} T_{\mu\nu} T_{\alpha\beta}
\]

References

1. S. Kopp et al. Dalitz analysis of the decay 

2. Hartmund Pilkuhn.

Table 1: Spin Factors.
Spin factors 4-body

Table B.1: respectively. If no angular momentum is specified, the lowest angular momentum state compatible

<table>
<thead>
<tr>
<th>Number</th>
<th>Decay chain</th>
<th>Spin amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$D \to (P_{1} P_{2})$, $P \to (S_{1} P_{2})$, $S \to (P_{1} S_{2})$</td>
<td>$L^{(1)\alpha}<em>{\mu}(P) L^{\alpha}</em>{(1\alpha)}(V)$</td>
</tr>
<tr>
<td>2</td>
<td>$D \to (P_{1} P_{2})$, $P \to (V_{1} P_{2})$, $V \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(D) P^{\alpha}</em>{(1\alpha)}(A) L^{(1\alpha)}_{(1\alpha)}(V)$</td>
</tr>
<tr>
<td>3</td>
<td>$D \to (A_{1} P_{1})$, $A \to (V P_{2})$, $V \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(D) L^{(2\alpha)}</em>{(1\alpha)}(A) L^{(1\alpha)}_{(1\alpha)}(V)$</td>
</tr>
<tr>
<td>4</td>
<td>$D \to (A_{1} P_{1})$, $A[D] \to (P_{2} V)$, $V \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(D) L^{(2\alpha)}</em>{(1\alpha)}(A) L^{(1\alpha)}_{(1\alpha)}(V)$</td>
</tr>
<tr>
<td>5</td>
<td>$D \to (A_{1} P_{1})$, $A \to (S_{1} P_{2})$, $S \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(D) L^{(2\alpha)}</em>{(1\alpha)}(A)$</td>
</tr>
<tr>
<td>6</td>
<td>$D \to (A_{1} P_{1})$, $A \to (T P_{2})$, $T \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(D) L^{(2\alpha)}</em>{(1\alpha)}(T)$</td>
</tr>
<tr>
<td>7</td>
<td>$D \to (V_{1} P_{2})$, $V_{1} \to (V_{2} P_{3})$, $V_{2} \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(D) P^{\alpha}</em>{(1\alpha)}(V_{1}) \epsilon_{\alpha \beta \gamma \delta} L_{(1\alpha)}^{(2\alpha)}(V_{1}) p_{1}^{\alpha} L_{(1\alpha)}^{(3\alpha)}(V_{2})$</td>
</tr>
<tr>
<td>8</td>
<td>$D \to (PT P_{1})$, $PT \to (V P_{2})$, $V \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) L^{\alpha}</em>{(2\alpha)}(PT) L^{(1\alpha)}<em>{(1\alpha)}(PT) L^{(1\alpha)}</em>{(1\alpha)}(V)$</td>
</tr>
<tr>
<td>9</td>
<td>$D \to (PT P_{1})$, $PT \to (S P_{2})$, $S \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) L^{\alpha}</em>{(2\alpha)}(PT) p_{1}^{\alpha}$</td>
</tr>
<tr>
<td>10</td>
<td>$D \to (PT P_{1})$, $PT \to (T P_{2})$, $T \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) L^{\alpha}</em>{(2\alpha)}(PT) L^{(1\alpha)}_{(1\alpha)}(T)$</td>
</tr>
<tr>
<td>11</td>
<td>$D \to (T P_{1})$, $T \to (V P_{2})$, $V \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) P^{\alpha}</em>{(1\alpha)}(T) \epsilon_{\alpha \beta \gamma \delta} L_{(2\alpha)}^{(3\alpha)}(T) p_{1}^{\alpha} p_{2}^{\alpha} L_{(1\alpha)}^{(3\alpha)}(V)$</td>
</tr>
<tr>
<td>12</td>
<td>$D \to (T_{1} P_{1})$, $T_{1} \to (T_{2} P_{2})$, $T_{2} \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) P^{\alpha}</em>{(2\alpha)}(T_{1}) \epsilon_{\alpha \beta \gamma \delta} L_{(2\alpha)}^{(3\alpha)}(T_{1}) p_{1}^{\alpha} L_{(2\alpha)}^{(3\alpha)}(T_{2})$</td>
</tr>
<tr>
<td>13</td>
<td>$D \to (S_{1} S_{2})$, $S_{1} \to (P_{1} P_{2})$, $S_{2} \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) L^{\alpha}</em>{(2\alpha)}(S_{1}) L^{(1\alpha)}<em>{(1\alpha)}(S</em>{1})$</td>
</tr>
<tr>
<td>14</td>
<td>$D \to (V S)$, $V \to (P_{1} P_{2})$, $S \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(D) L^{\alpha}</em>{(1\alpha)}(V)$</td>
</tr>
<tr>
<td>15</td>
<td>$D \to (V_{1} V_{2})$, $V_{1} \to (P_{1} P_{2})$, $V_{2} \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(V</em>{1}) L^{\alpha}<em>{(1\alpha)}(V</em>{2})$</td>
</tr>
<tr>
<td>16</td>
<td>$D[P^\mu] \to (V_{1} V_{2})$, $V_{1} \to (P_{1} P_{2})$, $V_{2} \to (P_{1} P_{3})$</td>
<td>$\epsilon_{\alpha \beta \gamma \delta} L^{(2\alpha)}<em>{(1\alpha)}(D) L</em>{(1\alpha)}^{(2\alpha)}(V_{1}) L_{(1\alpha)}^{(3\alpha)}(V_{2}) p_{D}^\mu$</td>
</tr>
<tr>
<td>17</td>
<td>$D[D] \to (V_{1} V_{2})$, $V_{1} \to (P_{1} P_{2})$, $V_{2} \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) L^{\alpha}</em>{(2\alpha)}(V_{1}) L_{(1\alpha)}^{(3\alpha)}(V_{2})$</td>
</tr>
<tr>
<td>18</td>
<td>$D \to (T S)$, $T \to (P_{1} P_{2})$, $S \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) L^{\alpha}</em>{(2\alpha)}(T)$</td>
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<td>19</td>
<td>$D \to (V T)$, $T \to (P_{1} P_{2})$, $V \to (P_{1} P_{3})$</td>
<td>$L^{(1)\alpha}<em>{\alpha}(D) L</em>{(1\alpha)}^{(3\alpha)}(T)$</td>
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<td>20</td>
<td>$D[D] \to (T V)$, $T \to (P_{1} P_{2})$, $V \to (P_{1} P_{3})$</td>
<td>$L^{(2\alpha)}<em>{\alpha}(D) L</em>{(2\alpha)}^{(3\alpha)}(T) p_{D}^\mu$</td>
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<td>$\epsilon_{\alpha \beta \gamma \delta} L_{(1\alpha)}^{(2\alpha)}(D) L_{(2\alpha)}^{(3\alpha)}(T_{1}) L_{(2\alpha)}^{(3\alpha)}(T_{2}) p_{D}^\mu$</td>
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<td>$D[D] \to (T_{1} T_{2})$, $T_{1} \to (P_{1} P_{2})$, $T_{2} \to (P_{1} P_{3})$</td>
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</tr>
</tbody>
</table>

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... and more

Jonas Rademacker
How to Build an Amplitude Model
BELLE II Physics Week, 3 Dec 2020
Mass dependent width (ignoring ang. mom)

\[ \Gamma(m_{23}) = \Gamma_0 \frac{(q_{23}/m_{23})}{(q_0/m_R)} \frac{B_L(q_{23})}{B_L(q_0)} \]

dashed: fixed width

solid: mass dependent width
Breit-Wigner with angular momentum effects (only)

\[ \Gamma(m_{23}) = \Gamma_0 \left( \frac{q_{23}/m_{23}}{q_0/m_R} \right) \frac{B_L(q_{23})}{B_L(q_0)} \]

no 2-body phase space
Mass dependent width ($a_1$)
Mass dependent width ($a_1$)

\[ a_1(1270)^+ [S] \rightarrow \pi^+ (\rho(770) \rightarrow \pi^+ \pi^-) \]

\[ a_1(1270)^+ [D] \rightarrow \pi^+ (\rho(770) \rightarrow \pi^+ \pi^-) \]
Mass dependent width (a1)
4-body amplitude analysis $D^o \rightarrow \pi \pi \pi \pi$ (CLEO data)

$D \rightarrow \pi^- a_1(1260)^+$
$D \rightarrow \pi^- \pi (1300)^+$
$D \rightarrow \pi^- a_1(1640)^+$
others

$\chi^2/\nu = 1.4$

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### C. Considered Decay Chains

<table>
<thead>
<tr>
<th>Decay channel</th>
<th>D(^0) → π(^-)</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(D^0 \rightarrow \pi^-)</td>
<td>(a_1(1260)^+) → (\pi^+ \sigma)</td>
<td>(S, D)</td>
<td>(a_1(1260)^+) → (\pi^+ f_0(980))</td>
<td>(a_1(1260)^+) → (\pi^+ f_2(1270))</td>
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</tr>
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<td>(\pi(1300)^+) → (\pi^+ \rho(770)^0)</td>
</tr>
</tbody>
</table>
Table C.1: The various decay channels considered in the model building are listed in Tables C.1 and C.2. For cascade components considered

\[ D \rightarrow \pi \pi \pi \text{ components considered} \]

\begin{align*}
D^0 & \rightarrow \rho(770)^0 \sigma \\
D^0 & \rightarrow \rho(770)^0 f_0(980) \\
D^0 & \rightarrow \rho(770)^0 f_2(1270) \\
D^0 & \rightarrow \rho(1450)^0 \sigma \\
D^0[S, P, D] & \rightarrow (\pi \pi)_P (\pi \pi)_P \\
D^0[S, P, D] & \rightarrow \rho(770)^0 (\pi \pi)_P \\
D^0[S, P, D] & \rightarrow \rho(770)^0 \rho(770)^0 \\
D^0[S, P, D] & \rightarrow \rho(770)^0 \omega(782)^0 \\
D^0[S, P, D] & \rightarrow \omega(782)^0 \omega(782)^0 \\
D^0[S, P, D] & \rightarrow \rho(1450)^0 (\pi \pi)_P \\
D^0[S, P, D] & \rightarrow \rho(1450)^0 \rho(1450)^0 \\
D^0 & \rightarrow f_2(1270) \sigma \\
D^0 & \rightarrow f_2(1270) f_0(980) \\
D^0[P, D] & \rightarrow f_2(1270) \rho(770)^0 \\
D^0[S, P, D] & \rightarrow f_2(1270) f_2(1270) \\
\end{align*}

\[ \text{JHEP 05 (2017) 143} \]
### Table C.1: Considered Decay Chains

<table>
<thead>
<tr>
<th>Decay channel</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^0 \rightarrow \pi^- a_1(1260)^+ \rightarrow \pi^+ \sigma$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \pi^- a_1(1260)^+[S, D] \rightarrow \pi^+ \rho(770)^0$</td>
<td></td>
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<tr>
<td>$D^0 \rightarrow \pi^- a_1(1260)^+[S, D] \rightarrow \pi^+ \rho(1450)^0$</td>
<td></td>
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<td></td>
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<tr>
<td>$D^0 \rightarrow \pi^- (1300)^+ \rightarrow \pi^+ (\pi^+ \pi^-)_P$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \pi^- a_2(1320)^+ \rightarrow \pi^+ \rho(770)^0$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \pi^- a_2(1320)^+ \rightarrow \pi^+ f_2(1270)$</td>
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<tr>
<td>$D^0 \rightarrow \pi^- \pi^+(1460)^+ \rightarrow \pi^+ \rho(770)^0$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \pi^- a_1(1640)^+ \rightarrow \pi^+ \sigma$</td>
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<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow (\pi \pi)_S (\pi \pi)_S$</td>
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</tr>
<tr>
<td>$D^0 \rightarrow \sigma \sigma$</td>
<td></td>
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</tr>
<tr>
<td>$D^0 \rightarrow \sigma f_0(1370)$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow f_0(980) f_0(980)$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow f_0(1370) f_0(1370)$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \rho(770)^0 \sigma$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \rho(770)^0 f_0(980)$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \rho(770)^0 f_0(1370)$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \rho(1450)^0 \sigma$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \rho(1450)^0 f_0(980)$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \rho(1450)^0 f_0(1370)$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \rho(1450)^0 \rho(1450)^0$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow \rho(1450)^0 \rho(1450)^0 \rho(1450)^0$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow f_2(1270) \sigma$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow f_2(1270) \rho(770)^0$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow f_2(1270) \rho(1450)^0$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow f_2(1270) f_0(980)$</td>
<td></td>
</tr>
<tr>
<td>$D^0 \rightarrow f_2(1270) f_0(1370)$</td>
<td></td>
</tr>
</tbody>
</table>
Lasso*, illustration by Philippe d’Argent

- Generated: pdf = 1 + x
- Fitted pdf = 1 + \( \sum_{i=1}^{10} c_i x^i \)
- \( -2 \cdot \log(L) \rightarrow -2 \cdot \log(L) + \lambda \cdot \sum_i |c_i| \)

\[ \lambda = 0.0 \]

\[
\begin{align*}
    c_1 &= -0.9500 \pm 2.1580 \\
    c_2 &= 10.6545 \pm 7.1052 \\
    c_3 &= -6.9912 \pm 11.8217 \\
    c_4 &= -10.2646 \pm 13.3412 \\
    c_5 &= -3.4468 \pm 14.7392 \\
    c_6 &= 5.2567 \pm 15.4221 \\
    c_7 &= 10.3363 \pm 16.0175 \\
    c_8 &= 9.1735 \pm 16.8335 \\
    c_9 &= 0.9337 \pm 16.6720 \\
    c_{10} &= -14.1801 \pm 12.4028 
\end{align*}
\]

*) least absolute shrinkage and selection operator
Lasso*, illustration by Philippe d’Argent

- Generated: $pdf = 1 + x$
- Fitted $pdf = 1 + \sum_{i=1}^{10} c_i x^i$
- $-2 \cdot \log(L) \rightarrow -2 \cdot \log(L) + \lambda \cdot \sum_i |c_i|$

\[\lambda = 0.0\]

- $c_1 = -0.9500 \pm 2.1580$
- $c_2 = 10.6545 \pm 7.1052$
- $c_3 = -6.9912 \pm 11.8217$
- $c_4 = -10.2646 \pm 13.3412$
- $c_5 = -3.4468 \pm 14.7392$
- $c_6 = 5.2567 \pm 15.4221$
- $c_7 = 10.3363 \pm 16.0175$
- $c_8 = 9.1735 \pm 16.8335$
- $c_9 = 0.9337 \pm 16.6720$
- $c_{10} = -14.1801 \pm 12.4028$

*) least absolute shrinkage and selection operator
LASSO illustration (by Philippe d’Argent)

How to choose $\lambda$?

- $\text{BIC}(\lambda) = -2 \cdot \log(L) + r \cdot \log(N_{\text{events}})$
  
  $r = \text{Number of parameters with: } |c_i| > \text{threshold (1%)}$

- Balances gain in fit quality vs. complexity

- Optimal value $\lambda \approx 4$
Lasso in $D^0 \rightarrow \pi^+ \pi^- \pi^+ \pi^-$ amplitude analysis

\[- 2 \log \mathcal{L} \rightarrow - 2 \log \mathcal{L} + \lambda \sum_i \sqrt{\int |a_i A_i(x)|^2 \, d\Phi_4}\]

How to choose $\lambda$?
- Now for amplitude fit for real data
- Optimal value $\lambda \approx 30$

\[\Delta \text{BIC} \quad \text{d'}Argent et al} \]
Lasso’s limits

Any two of these:

\[ D \to (\pi \pi)_S (\pi \pi)_S \]
\[ D \to (\pi \pi)_S \sigma \]
\[ D \to \sigma \sigma \]
\[ D \to \sigma f_0(1370) \]
\[ D \to f_0(1370) f_0(1370) \]

tend to produce huge interference effects; only one of this group is used at a time.
Lasso’s limits

Any two of these:

\[
D \rightarrow (\pi \pi)_P (\pi \pi)_P \\
D \rightarrow (\pi \pi)_P \rho(1450)^0 \\
D \rightarrow \rho(1450)^0 \rho(1450)^0
\]

tend to produce huge interference effects; only one of this group is used at a time.
Lasso’s limits

Including

\[ D \rightarrow \pi[\pi(1300) \rightarrow \pi(\pi \pi)_P] \]

leads to

\[ D \rightarrow \rho(770)^0 \rho(770)^0 \]

D-wave that’s much larger than the \( \rho \rho \) S-wave, with a very large destructive interference. Although LASSO seems to like this, we did not, and excluded this component.
Lasso in $D^0 \rightarrow \pi^+ \pi^- \pi^+ \pi^-$ amplitude analysis

16 amplitudes selected
$D \rightarrow \pi^- a_1(1260)^+$
$D \rightarrow \pi^- \pi(1300)^+$
$D \rightarrow \pi^- a_1(1640)^+$
others
$\chi^2_{5D}/\nu = 1.4$

[JHEP 05 (2017) 143]
Lasso in $D^0 \rightarrow \pi^+ \pi^- \pi^+ \pi^-$ amplitude analysis

**Fit fraction:** \( F_i = \frac{\int |r_i A_i|^2 d\Phi}{\int \sum_j r_j e^{i\delta_j} A_j |d\Phi} \)

<table>
<thead>
<tr>
<th>Decay channel</th>
<th>( F_i (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D^0 \rightarrow \pi^- [a_1(1260)^+ \rightarrow \pi^+ \rho(770)^0] )</td>
<td>38.1 ± 2.3 ± 3.2 ± 1.7</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^- [a_1(1260)^+ \rightarrow \pi^+ \sigma] )</td>
<td>10.2 ± 1.4 ± 2.1 ± 2.5</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^+ [a_1(1260)^- \rightarrow \pi^- \rho(770)^0] )</td>
<td>3.1 ± 0.6 ± 0.5 ± 0.9</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^+ [a_1(1260)^- \rightarrow \pi^- \sigma] )</td>
<td>0.8 ± 0.2 ± 0.1 ± 0.4</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^- [\pi(1300)^+ \rightarrow \pi^+ \sigma] )</td>
<td>6.8 ± 0.9 ± 1.5 ± 3.1</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^+ [\pi(1300)^- \rightarrow \pi^- \sigma] )</td>
<td>3.0 ± 0.6 ± 2.0 ± 2.0</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^- [a_1(1640)^+ [D] \rightarrow \pi^+ \rho(770)^0] )</td>
<td>4.2 ± 0.6 ± 0.9 ± 1.8</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^- [a_1(1640)^+ \rightarrow \pi^+ \sigma] )</td>
<td>2.4 ± 0.7 ± 1.1 ± 1.3</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^- [\pi_2(1670)^+ \rightarrow \pi^+ f_2(1270)] )</td>
<td>2.7 ± 0.6 ± 0.7 ± 0.9</td>
</tr>
<tr>
<td>( D^0 \rightarrow \pi^- [\pi_2(1670)^+ \rightarrow \pi^+ \sigma] )</td>
<td>3.5 ± 0.6 ± 0.8 ± 0.9</td>
</tr>
<tr>
<td>( D^0 \rightarrow \sigma f_0(1370) )</td>
<td>21.2 ± 1.8 ± 4.2 ± 5.2</td>
</tr>
<tr>
<td>( D^0 \rightarrow \sigma \rho(770)^0 )</td>
<td>6.6 ± 1.0 ± 1.2 ± 3.0</td>
</tr>
<tr>
<td>( D^0[S] \rightarrow \rho(770)^0 \rho(770)^0 )</td>
<td>2.4 ± 0.7 ± 1.1 ± 1.0</td>
</tr>
<tr>
<td>( D^0[P] \rightarrow \rho(770)^0 \rho(770)^0 )</td>
<td>7.0 ± 0.5 ± 1.6 ± 0.3</td>
</tr>
<tr>
<td>( D^0[D] \rightarrow \rho(770)^0 \rho(770)^0 )</td>
<td>8.2 ± 1.0 ± 1.7 ± 3.5</td>
</tr>
<tr>
<td>( D^0 \rightarrow f_2(1270) f_2(1270) )</td>
<td>2.1 ± 0.5 ± 0.3 ± 2.3</td>
</tr>
<tr>
<td>Sum</td>
<td>122.0 ± 4.0 ± 6.4 ± 7.6</td>
</tr>
</tbody>
</table>

[JHEP 05 (2017) 143]
Alternative models

Table D.2: Fit fractions in percent for each component of various alternative models for $D^0 \rightarrow \pi^+\pi^-\pi^+\pi^-$ based on fit quality. Resonance parameters, $F_{\pi^+}^4$ and $\chi^2/\nu$ are also given. The uncertainties are statistical only.

<table>
<thead>
<tr>
<th>Decay mode</th>
<th>Alt. 1</th>
<th>Alt. 2</th>
<th>Alt. 3</th>
<th>Alt. 4</th>
<th>Alt. 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^0 \rightarrow \pi^-$</td>
<td>37.1 ± 2.3</td>
<td>38.3 ± 2.4</td>
<td>35.2 ± 2.6</td>
<td>38.4 ± 2.5</td>
<td>35.7 ± 2.7</td>
</tr>
<tr>
<td>$a_1(1260)^+ \rightarrow \pi^+ \rho(770)^0$</td>
<td>11.3 ± 1.0</td>
<td>9.8 ± 1.2</td>
<td>9.4 ± 1.2</td>
<td>11.6 ± 1.4</td>
<td>11.4 ± 1.7</td>
</tr>
<tr>
<td>$a_1(1260)^- \rightarrow \pi^+ \rho(770)^0$</td>
<td>2.1 ± 0.5</td>
<td>3.3 ± 0.6</td>
<td>3.7 ± 0.7</td>
<td>3.1 ± 0.6</td>
<td>4.1 ± 0.7</td>
</tr>
<tr>
<td>$a_1(1260)^- \rightarrow \pi^- \rho(770)^0$</td>
<td>0.6 ± 0.2</td>
<td>0.9 ± 0.2</td>
<td>1.0 ± 0.3</td>
<td>0.9 ± 0.2</td>
<td>1.3 ± 0.3</td>
</tr>
<tr>
<td>$\pi(1300)^+ \rightarrow \pi^+ (\pi^+\pi^-)^p$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>6.4 ± 1.3</td>
</tr>
<tr>
<td>$\pi(1300)^+ \rightarrow \pi^+ (\pi^+\pi^-)^p$</td>
<td>8.1 ± 1.0</td>
<td>8.6 ± 1.4</td>
<td>6.0 ± 1.0</td>
<td>7.7 ± 1.6</td>
<td>4.3 ± 1.1</td>
</tr>
<tr>
<td>$\pi(1300)^- \rightarrow \pi^+ (\pi^+\pi^-)^p$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.5 ± 0.5</td>
</tr>
<tr>
<td>$\pi(1300)^+ \rightarrow \pi^- (\pi^+\pi^-)^p$</td>
<td>4.3 ± 0.9</td>
<td>4.0 ± 1.5</td>
<td>6.8 ± 1.6</td>
<td>4.9 ± 1.6</td>
<td>1.7 ± 0.4</td>
</tr>
<tr>
<td>$a_1(1640)^+ \rightarrow \pi^+ \rho(770)^0$</td>
<td>2.7 ± 0.9</td>
<td>4.5 ± 1.5</td>
<td>3.9 ± 1.6</td>
<td>5.2 ± 1.1</td>
<td>3.7 ± 1.8</td>
</tr>
<tr>
<td>$a_1(1640)^- \rightarrow \pi^- \rho(770)^0$</td>
<td>3.2 ± 1.3</td>
<td>1.4 ± 0.5</td>
<td>2.4 ± 1.0</td>
<td>3.0 ± 0.9</td>
<td>1.2 ± 0.7</td>
</tr>
<tr>
<td>$\pi(1270)^+ \rightarrow \pi^+ f_2(2170)$</td>
<td>1.8 ± 0.5</td>
<td>0.6 ± 0.2</td>
<td>1.2 ± 0.4</td>
<td>1.7 ± 0.5</td>
<td>1.6 ± 0.4</td>
</tr>
<tr>
<td>$\pi(1270)^- \rightarrow \pi^- f_2(2170)$</td>
<td>2.7 ± 0.5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\pi(1270)^+ \rightarrow \pi^- \rho(770)^0$</td>
<td>2.1 ± 0.4</td>
<td>3.9 ± 0.6</td>
<td>3.3 ± 0.6</td>
<td>3.8 ± 0.6</td>
<td>3.5 ± 0.6</td>
</tr>
<tr>
<td>$\pi(1270)^- \rightarrow \pi^+ \rho(770)^0$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\sigma f_0(980)$</td>
<td>20.7 ± 2.2</td>
<td>19.3 ± 2.4</td>
<td>21.3 ± 2.4</td>
<td>21.8 ± 2.5</td>
<td>20.4 ± 2.1</td>
</tr>
<tr>
<td>$\rho(770)^0$</td>
<td>5.5 ± 1.0</td>
<td>8.7 ± 1.2</td>
<td>8.7 ± 1.4</td>
<td>-</td>
<td>4.8 ± 1.2</td>
</tr>
<tr>
<td>$\sigma f_0(1370)$</td>
<td>-</td>
<td>-</td>
<td>3.6 ± 0.8</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\rho(770)^0$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5.8 ± 1.0</td>
<td>-</td>
</tr>
<tr>
<td>$\rho(770)^0$</td>
<td>-</td>
<td>1.5 ± 0.4</td>
<td>0.8 ± 0.4</td>
<td>1.2 ± 0.4</td>
<td>0.9 ± 0.4</td>
</tr>
<tr>
<td>$\rho(770)^0$</td>
<td>7.3 ± 0.5</td>
<td>6.8 ± 0.5</td>
<td>6.9 ± 0.5</td>
<td>6.8 ± 0.5</td>
<td>6.4 ± 0.5</td>
</tr>
<tr>
<td>$\rho(770)^0$</td>
<td>10.4 ± 0.9</td>
<td>8.3 ± 1.0</td>
<td>11.4 ± 1.4</td>
<td>10.9 ± 1.2</td>
<td>16.0 ± 2.1</td>
</tr>
<tr>
<td>$\rho(770)^0$</td>
<td>2.5 ± 0.5</td>
<td>-</td>
<td>1.2 ± 0.3</td>
<td>1.4 ± 0.4</td>
<td>1.1 ± 0.3</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>122 ± 4</td>
<td>120 ± 3</td>
<td>127 ± 4</td>
<td>128 ± 4</td>
<td>127 ± 6</td>
</tr>
<tr>
<td>$m_{a_1(1260)}$ (MeV/c²)</td>
<td>1198 ± 8</td>
<td>1220 ± 8</td>
<td>1213 ± 9</td>
<td>1215 ± 8</td>
<td>1231 ± 9</td>
</tr>
<tr>
<td>$G_{a_1(1260)}$ (MeV)</td>
<td>429 ± 24</td>
<td>408 ± 23</td>
<td>434 ± 24</td>
<td>420 ± 24</td>
<td>459 ± 25</td>
</tr>
<tr>
<td>$m_{\pi(1300)}$ (MeV/c²)</td>
<td>1110 ± 17</td>
<td>1079 ± 25</td>
<td>1075 ± 22</td>
<td>1077 ± 36</td>
<td>1180 ± 15</td>
</tr>
<tr>
<td>$G_{\pi(1300)}$ (MeV)</td>
<td>314 ± 39</td>
<td>347 ± 40</td>
<td>330 ± 39</td>
<td>377 ± 41</td>
<td>297 ± 36</td>
</tr>
<tr>
<td>$m_{a_1(1640)}$ (MeV/c²)</td>
<td>1694 ± 19</td>
<td>1681 ± 18</td>
<td>1672 ± 22</td>
<td>1686 ± 18</td>
<td>1644 ± 16</td>
</tr>
<tr>
<td>$G_{a_1(1640)}$ (MeV)</td>
<td>177 ± 45</td>
<td>171 ± 36</td>
<td>250 ± 59</td>
<td>209 ± 28</td>
<td>222 ± 56</td>
</tr>
<tr>
<td>$\chi^2/\nu$</td>
<td>1.50</td>
<td>1.42</td>
<td>1.43</td>
<td>1.50</td>
<td>1.33</td>
</tr>
<tr>
<td>$\nu$</td>
<td>221</td>
<td>223</td>
<td>219</td>
<td>221</td>
<td>219</td>
</tr>
<tr>
<td>$F_{\pi^+}^4$ (%)</td>
<td>71.7 ± 0.9</td>
<td>72.9 ± 0.9</td>
<td>73.0 ± 0.9</td>
<td>73.3 ± 0.9</td>
<td>73.5 ± 0.9</td>
</tr>
</tbody>
</table>
### Alternative models

Table D.1: Fit fractions in percent for each component of specific alternative models for $D^0 \rightarrow \pi^+\pi^-\pi^+\pi^-$. Resonance parameters, $F_{4s}^{\pi}$ and $\chi^2/\nu$ are also given. The uncertainties are statistical only.
a(1260), model-independent compared to B-W
a(1640), model-independent compared to B-W
π(1300), model-independent compared to B-W

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Lasso in $D^0 \rightarrow K^+K^-\pi^+\pi^-$ amplitude analysis

Stage 1: components with 2 resonances

Stage 2: components with 1 resonance

At each stage, some manual interference necessary
Lasso’s limits

• Lasso usually needs adult supervision. Some amplitudes, for example, interfere so much with each other that they cannot be in the same model.

• Lasso can only remove amplitudes, not add them - so in principle all possible amplitudes need to be in the same fit. As the number of plausible contributions increases (4-body decays with very large dataset), Lasso becomes unviable.
$D^0 \rightarrow K^+ K^- \pi^+ \pi^-$

**CLEO-c data**

**LHCb-data**

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$K_S$ veto

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\(D^0 \rightarrow K^+ K^- \pi^+ \pi^-\)

CLEO-c data

LHCb-data

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\(K_S\) veto

JHEP 1902 (2019) 126
$D^0 \rightarrow K^+ K^- \pi^+ \pi^-$

CLEO-c data

LHCb-data

$\phi(1020)$

$K^0^*$

$K_S$ veto

$\phi(1020)$

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$D^0 \rightarrow K^+ K^- \pi^+ \pi^-$

CLEO-c data

LHCb-data

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Approaches where components are added

Start with $D^0 \rightarrow \phi(1020)(\rho - \omega)^0$  $D^0 \rightarrow K^*(892)^0 \overline{K}^*(892)^0$ (each in S, P, D wave)
Then add the component from the “pool” that decreases $\chi^2$ maximally, until sum of fit-fractions (i.e. the amount of destructive interference) blows up.

![Graphs showing evolution of $\chi^2$/ndf and sum of fit fractions during model building procedure](image)

Figure 4.8 – Evolution of the $\chi^2$/ndf and the sum of the fit fractions during the model building procedure. The horizontal axis shows the number of added amplitudes.

JHEP 1902 (2019) 126 and Maxime Schubiger, thesis
It’s a treacherous mountainous landscape
It’s a treacherous mountainous landscape
It’s a treacherous mountainous landscape
It’s a treacherous mountainous landscape
It’s a treacherous mountainous landscape
It’s a treacherous mountainous landscape
\[ D^0, \overline{D}^0 \rightarrow K^- \pi^+ \pi^- \pi^+ \]

Taking multiple path

1. Take a model and a set of ~100 components. Perform a fit to the data using this model adding one of these components.

2. If adding the component improves the $\chi^2$ per degree of freedom by at least 0.02, then retain the model for further consideration.

3. On the first iteration, restrict the pool of decay chains that are added to the model to those 40 contributions that give the largest improvements to the fit.

4. Reiterate the model-building procedure, using the 15 models with the best fit quality from step 2 as starting points. Finish the procedure if no model has improved significantly.

$D^0 \rightarrow K^- \pi^+ \pi^- \pi^+$

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Fit Fraction [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[K^+(892)\rho(770)^0]_{L=0}$</td>
<td>$7.34 \pm 0.08 \pm 0.47$</td>
</tr>
<tr>
<td>$[K^+(892)\rho(770)^0]_{L=1}$</td>
<td>$6.03 \pm 0.05 \pm 0.25$</td>
</tr>
<tr>
<td>$[K^+(892)\rho(770)^0]_{L=2}$</td>
<td>$8.47 \pm 0.09 \pm 0.67$</td>
</tr>
<tr>
<td>$[\rho(1450)K^+(892)]_{L=0}$</td>
<td>$0.61 \pm 0.04 \pm 0.17$</td>
</tr>
<tr>
<td>$[\rho(1450)K^+(892)]_{L=1}$</td>
<td>$1.98 \pm 0.03 \pm 0.33$</td>
</tr>
<tr>
<td>$[\rho(1450)K^+(892)]_{L=2}$</td>
<td>$0.46 \pm 0.03 \pm 0.15$</td>
</tr>
</tbody>
</table>

$\rho(770)^0 [K^-(\pi^+)]_{L=0}$  
$\alpha_{3/2}$ $K^+(892)\pi^+ \pi^-$ $\beta_1$

$[K^-(\pi^+)]_{L=0}$  
$\alpha_{3/2}$ $\alpha_{K^+P}$ $\beta_1$ $f_{\pi \pi}$

Sum of Fit Fractions  
$\chi^2/\nu$  
$98.29 \pm 0.37 \pm 0.84$  
$40483/32701 = 1.23\delta$

![Graphs showing distributions for six invariant-mass observables in the RS decay](image-url)
Word’s of wisdom…

Tim Evans: “I don't think its very easy to separate the model-building from the descriptions of individual components (the other reason why the MIPWA is so useful); although the model-building gave us more-or-less the "correct" components anyway, the biggest improvements in the fit quality […] came from improving the description of the components (with K-matrices, or accounting for multi-body running widths)”
$B_s \rightarrow \bar{D}K^-\pi^+$

Figure 3: Distribution of $B^0_s \rightarrow \bar{D}K^-\pi^+$ candidates in the signal region over (a) the Dalitz plot and (b) the square Dalitz plot defined in Eq. (19). The effect of the $D^0$ veto can be seen as an unpopulated horizontal (curved) band in the (square) Dalitz plot.

The resonance dynamics are contained within the $F_j$ terms, which are composed of invariant mass and angular distributions and are normalised such that the integral over the Dalitz plot of the squared magnitude of each term is unity.

For example, for a $D^0K^-$ resonance

$$F_j = R_j m^2(D^0K^-) X(|\tilde{p}| r_{BW}) X(|\tilde{q}| r_{BW}) T(\tilde{p}, \tilde{q}),$$

where the functions $R$, $X$ and $T$ described below depend on parameters of the resonance such as its spin $L$, pole mass $m_0$ and width $\Gamma_0$. In the case of a $D^0K^-$ resonance, the $\pi^+$ is referred to as the "bachelor" particle. Since the $B^0_s$ meson has zero spin, $L$ is equivalently the orbital angular momentum between the resonance and the bachelor.

In Eq. (3), the function $R_j m^2(D^0K^-)$ is the resonance mass term (given e.g. by a Breit–Wigner shape — the detailed forms for each of the resonance shapes used in the model are described below), while $\tilde{p}$ and $\tilde{q}$ are the momenta of the bachelor particle and one of the resonance daughters, respectively, both evaluated in the rest frame of the resonance. The terms $X(z)$, where $z = |\tilde{q}| r_{BW}$ or $|\tilde{p}| r_{BW}$, are Blatt–Weisskopf barrier form factors [27], and are given by

$$X(z) = \begin{cases} 1 & \text{for } L = 0 \\ s_1 + z^2 s_0 & \text{for } L = 1 \end{cases},$$

where $s_1 = s_0 = 1$. These form factors are used to describe the phase space suppression effects due to the finite width of the resonance.
Fit Projections

LHCb: PRL 113 (2014) 162001
LHCb: PRD 90 (2014) 072003

Jonas Rademacker

How to Build an Amplitude Model

BELLE II Physics Week, 3 Dec 2020
The $D^*_sJ(2860)$


The considered sources of systematic uncertainty are divided into two main categories: variation of the e from imperfect knowledge of: the relative amount of signal and background in the selected channel. The spin is 2. At higher masses, interpretation of the moments becomes more difficult. The D* resonance and (d) the square Dalitz plot defined in Eq. (19). The e candidates in the signal region over (a) the Dalitz plot and (b) the Dalitz plot with a logarithmic scale in (b), (d) and (e) the Dalitz plot with a logarithmic scale in (b), (d) and (e). The functions form factors [27], and are given by

\[
F_{LHCb}^{\pi}(m) = \frac{1}{\sqrt{\pi}} \frac{m^{3/2}}{m_0^{5/2}} e^{-m/m_0},
\]

where the functions form factors [27], and are given by

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\[
F_{LHCb}^{\pi}(m) = \frac{1}{\sqrt{\pi}} \frac{m^{3/2}}{m_0^{5/2}} e^{-m/m_0},
\]
Helicity angles and Dalitz plot variables

Helicity distributions of spin $L$ described by sum of Legendre polynomials of order $\leq 2L$.

Legendre Polynomials

Helicity distributions of spin L described by sum of Legendre polynomials of order $\leq 2L$.

$$\frac{d\Gamma}{d(\cos \theta)} = \sum_{n=1}^{n_{\text{max}}} a_n P_n(\cos \theta)$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} \frac{d\Gamma}{d\cos \theta} P_n(\cos \theta) d(\cos \theta)$$

$$\approx \frac{1}{N} \frac{2n+1}{2} \sum_{\text{events in slice}} P_n(\cos \theta) d(\cos \theta)$$

(for flat efficiency)
Legendre Polynomials

Helicity distributions of spin \( L \) described by sum of Legendre polynomials of order \( \leq 2L \).

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\frac{d\Gamma}{d(\cos\theta)} = \sum_{n=1}^{n_{\text{max}}} a_n P_n(\cos\theta)
\]

\[
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\]

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\]

(for flat efficiency)
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\]

\[
a_n = \frac{2n+1}{2} \int_{-1}^{1} d\cos \theta \frac{d\Gamma}{d\cos \theta} P_n(\cos \theta) d(\cos \theta)
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\[
\approx \frac{1}{N} \frac{2n+1}{2} \sum \text{events in slice} P_n(\cos \theta) d(\cos \theta)
\]

(for flat efficiency)

structure in \( K\pi \) dominated by spin
Legendre Polynomials

Helicity distributions of spin L described by sum of Legendre polynomials of order \( \leq 2L \).

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a_n = \frac{2n+1}{2} \int_{-1}^{1} \frac{d\Gamma}{d\cos \theta} P_n(\cos \theta) d(\cos \theta)
\]

\[
\approx \frac{1}{N} \frac{2n+1}{2} \sum \text{events in slice} P_n(\cos \theta) d(\cos \theta)
\]

(for flat efficiency)
Legendre Polynomials

\begin{align*}
\frac{p^{(1)}}{0.036\text{GeV/c}} &< 0.02 \\
\frac{p^{(2)}}{0.036\text{GeV/c}} &< 0.04 \\
\frac{p^{(3)}}{0.036\text{GeV/c}} &< 0.06 \\
\frac{p^{(4)}}{0.036\text{GeV/c}} &< 0.08 \\
\frac{p^{(5)}}{0.036\text{GeV/c}} &< 0.10 \\
\frac{p^{(6)}}{0.036\text{GeV/c}} &< 0.12 \\
\frac{p^{(7)}}{0.036\text{GeV/c}} &< 0.14
\end{align*}

\[ LHCb \]

\begin{align*}
\frac{m^{(1)^2}}{0.036\text{GeV/c}} &< 2.4 \\
\frac{m^{(2)^2}}{0.036\text{GeV/c}} &< 2.6 \\
\frac{m^{(3)^2}}{0.036\text{GeV/c}} &< 2.8 \\
\frac{m^{(4)^2}}{0.036\text{GeV/c}} &< 3.0 \\
\frac{m^{(5)^2}}{0.036\text{GeV/c}} &< 3.2 \\
\frac{m^{(6)^2}}{0.036\text{GeV/c}} &< 3.4 \\
\frac{m^{(7)^2}}{0.036\text{GeV/c}} &< 3.6
\end{align*}

\[ LHCb \]

\begin{align*}
\frac{m^{(1)^3}}{0.036\text{GeV/c}} &< 2.4 \\
\frac{m^{(2)^3}}{0.036\text{GeV/c}} &< 2.6 \\
\frac{m^{(3)^3}}{0.036\text{GeV/c}} &< 2.8 \\
\frac{m^{(4)^3}}{0.036\text{GeV/c}} &< 3.0 \\
\frac{m^{(5)^3}}{0.036\text{GeV/c}} &< 3.2 \\
\frac{m^{(6)^3}}{0.036\text{GeV/c}} &< 3.4 \\
\frac{m^{(7)^3}}{0.036\text{GeV/c}} &< 3.6
\end{align*}

\[ LHCb \]

\begin{align*}
\frac{m^{(1)^4}}{0.036\text{GeV/c}} &< 2.4 \\
\frac{m^{(2)^4}}{0.036\text{GeV/c}} &< 2.6 \\
\frac{m^{(3)^4}}{0.036\text{GeV/c}} &< 2.8 \\
\frac{m^{(4)^4}}{0.036\text{GeV/c}} &< 3.0 \\
\frac{m^{(5)^4}}{0.036\text{GeV/c}} &< 3.2 \\
\frac{m^{(6)^4}}{0.036\text{GeV/c}} &< 3.4 \\
\frac{m^{(7)^4}}{0.036\text{GeV/c}} &< 3.6
\end{align*}

\[ LHCb \]

\begin{align*}
\frac{m^{(1)^5}}{0.036\text{GeV/c}} &< 2.4 \\
\frac{m^{(2)^5}}{0.036\text{GeV/c}} &< 2.6 \\
\frac{m^{(3)^5}}{0.036\text{GeV/c}} &< 2.8 \\
\frac{m^{(4)^5}}{0.036\text{GeV/c}} &< 3.0 \\
\frac{m^{(5)^5}}{0.036\text{GeV/c}} &< 3.2 \\
\frac{m^{(6)^5}}{0.036\text{GeV/c}} &< 3.4 \\
\frac{m^{(7)^5}}{0.036\text{GeV/c}} &< 3.6
\end{align*}

\[ LHCb \]

\begin{align*}
\frac{m^{(1)^6}}{0.036\text{GeV/c}} &< 2.4 \\
\frac{m^{(2)^6}}{0.036\text{GeV/c}} &< 2.6 \\
\frac{m^{(3)^6}}{0.036\text{GeV/c}} &< 2.8 \\
\frac{m^{(4)^6}}{0.036\text{GeV/c}} &< 3.0 \\
\frac{m^{(5)^6}}{0.036\text{GeV/c}} &< 3.2 \\
\frac{m^{(6)^6}}{0.036\text{GeV/c}} &< 3.4 \\
\frac{m^{(7)^6}}{0.036\text{GeV/c}} &< 3.6
\end{align*}

\[ LHCb \]

\begin{align*}
\frac{m^{(1)^7}}{0.036\text{GeV/c}} &< 2.4 \\
\frac{m^{(2)^7}}{0.036\text{GeV/c}} &< 2.6 \\
\frac{m^{(3)^7}}{0.036\text{GeV/c}} &< 2.8 \\
\frac{m^{(4)^7}}{0.036\text{GeV/c}} &< 3.0 \\
\frac{m^{(5)^7}}{0.036\text{GeV/c}} &< 3.2 \\
\frac{m^{(6)^7}}{0.036\text{GeV/c}} &< 3.4 \\
\frac{m^{(7)^7}}{0.036\text{GeV/c}} &< 3.6
\end{align*}

\[ LHCb \]
First observation of a heavy flavoured spin-3 resonance, and the first time spin-3 particle has been seen in B decays
$B^+ \rightarrow D^+ D^- K^+$

Main aim: $\psi \rightarrow D^+ D^-$ spectroscopy, interesting in its own right, and in particular in the context of $B^+ \rightarrow K^+ \mu^+ \mu^-$ and related flavour anomalies.
$B^+ \rightarrow D^+ D^- K^+$

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$B^+ \rightarrow D^+ D^- K^+$

Table 4: Magnitude and phase of the complex coefficients in the amplitude model, together with fit fractions for each component. The quantities are reported after correction for fit biases.

Figure 10: Comparisons of the invariant-mass distributions of candidates in the decay $D^0 \rightarrow D^+ D^-$ in (a) $m(D^+ D^-)$ and (b) $m(D^- K^+)$ bins.

- $\psi(3770) \rightarrow D^+ D^-$
- $\chi_0(3930) \rightarrow D^+ D^-$
- $\chi_2(3930) \rightarrow D^+ D^-$
- $\psi(4040) \rightarrow D^+ D^-$
- $\psi(4160) \rightarrow D^+ D^-$
- $\psi(4415) \rightarrow D^+ D^-$
- $X_0(2900) \rightarrow D^- K^+$
- $X_1(2900) \rightarrow D^- K^+$
- Nonresonant
\[ B^+ \rightarrow D^+D^-K^+ \]

Figure 6: Run 2 data entering the amplitude fit, shown in the Dalitz plot and its projection onto the invariant-mass squared for each of the three pairs of the final-state particles.

8 Results

8.1 Model excluding D+K resonances

The data in Figs. 5 and 6 exhibit a striking excess at \( m^2(D^+K^+) \sim 8 \pm 0.25 \text{ GeV}^2/c^4 \), both in Run 1 and Run 2, which cannot be accounted for by introducing resonances only in the \( D^+D^- \) decay channel. To illustrate this, the first model presented excludes any resonant content from the \( D^+K^+ \) channel. The model includes the \((3770),(3930),(3930), (4040), (4160), \) and \((4415)\) resonances, which are necessary to describe structure in the \( m(D^+D^-) \) spectrum. An on-resonant component is included and described by an exponential S-wave lineshape in the \( D^+K^+ \) spectrum.

The Dalitz-plot projections from this fit are compared to the data in Fig. 7. Contributions from individual components are superimposed. The goodness of fit is quantified in Fig. 8, where the largest deviations are seen in the \( m^2(D^+K^+) \sim 8 \pm 0.25 \text{ GeV}^2/c^4 \) region of the Dalitz plot. To illustrate this more clearly, a comparison between data and the result of the fit is made in Fig. 9 after excluding low-mass charmonium resonances through the requirement \( m(D^+D^-) > 4 \text{ GeV}/c^2 \).

It is concluded that a satisfactory description of the data cannot be obtained without including one or more components that model structure in \( m(D^+K^+) \) explicitly.
First open charm tetraquark
Moreover, including such a state in the model, either by itself or together with a same spin in the same two-body combination, and with freely varying masses and widths, to be spin 3). The impact of these di

\[ m(D^0D^-) \text{[GeV}^2/c^4] \]

It is concluded that a satisfactory description of the data cannot be obtained without

\[ m(D^-K^+) \text{[GeV}^2/c^4] \]

As described in Sec. 7.1,

\[ m(D^0D^-) \text{[GeV}^2/c^4] \]

Candidates / (17.3 MeV/c^2)

\[ m(D^-K^+) \text{[GeV}^2/c^4] \]

Candidates / 0.027

\[ m(D^-K^+) \text{[GeV}^2/c^4] \]

\[ \cos(\theta(D^-K^+)) \]

\[ X_0(2900) \rightarrow D^-K^+ \]

\[ X_1(2900) \rightarrow D^-K^+ \]
$\chi_c J(3930)$ region

\[\chi_c J(3930) \rightarrow D^+ D^-\]

\[\chi_{c0}(3930) \rightarrow D^+ D^-\]

\[\chi_{c2}(3930) \rightarrow D^+ D^-\]
2-D pull

The goodness of fit is visualised using the binned normalised residual distribution in Fig. 13. The $\chi^2/\text{ndf}$ is $86.1/38.3 = 2.25$, where the number of degrees of freedom, ndf, is an effective value obtained from pseudoexperiments and only statistical uncertainties are considered. While an overall reasonable description of the data is achieved with the baseline model, there are regions of the Dalitz plot where significant imperfections remain. The largest contributions to the binned $\chi^2$ are at $(m^2(D^+D^-), m^2(D^-K^+)) \in (10.5 \text{ GeV}^2/c^4, 13.5 \text{ GeV}^2/c^4)$ and $(10.5 \text{ GeV}^2/c^4, 18.5 \text{ GeV}^2/c^4)$. The disagreement in the first of these regions can also be seen in the $D^+D^-$ helicity angle distribution at low $m^2(D^-K^+)$, shown in Fig. 14, which shows a clear asymmetry most likely originating from interference between the $(3770) P$-wave state and $S$-wave $D^+D^-$ structure. Since the baseline model has only very limited $S$ wave in this region, the asymmetry observed in data cannot be reproduced in the model. This disagreement can also be seen in some other projections, for example at high $m^2(D^-K^+)$ in the projection of the whole Dalitz plot (Fig. 10).

The second of the aforementioned regions of data-model disagreement corresponds to low values of $m^2(D^-K^+)$, $m^2(D^+D^-)$. No particular disagreement is seen in other projections of this region, and therefore it is not considered a source of concern. There does seem to be some disagreement at high $m^2(D^-K^+)$ values (Fig. 10), but this does not make a large contribution to the $\chi^2$ value. While the region around the $(4415)$ resonance does not appear to be perfectly modelled in the projection, it is probable that at least some of this is statistical, since a very sharp structure at $m^2(D^+D^-) \in 4\text{.47 GeV}^2/c^4$ seems unlikely to be physical.

In summary, while the baseline model does not perfectly reproduce the observed Dalitz-plot distribution, it gives the best description of the currently available data, with $LHCb$: arXiv:2011.09112 [hep-ex] (accepted by PRL) 
The missing bit
Legendre Polynomials

Helicity distributions of spin L described by sum of Legendre polynomials of order $\leq 2L$.

$$\frac{d\Gamma}{d(\cos \theta)} = \sum_{n=1}^{n_{\text{max}}} a_n P_n(\cos \theta)$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} \frac{d\Gamma}{d \cos \theta} P_n(\cos \theta) d(\cos \theta)$$

$$\approx \frac{1}{N} \frac{2n+1}{2} \sum_{\text{events in slice}} P_n(\cos \theta) d(\cos \theta)$$

(for flat efficiency)
Legendre Polynomials

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\]

\[
\approx \frac{1}{N} \frac{2n + 1}{2} \sum_{\text{events in slice}} P_n(\cos \theta) d(\cos \theta)
\]

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Helicity distributions of spin L described by sum of Legendre polynomials of order $\leq 2L$.

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$$\approx \frac{1}{N} \frac{2n + 1}{2} \sum_{\text{events in slice}} P_n(\cos \theta) d(\cos \theta)$$

(for flat efficiency)
$B^{\pm} \to \pi^{\pm} \pi^+ \pi^-$

$\rho$ P wave interferes with S-wave

$\Gamma(M \to f) - \Gamma(\bar{M} \to \bar{f}) = |\langle f | T | M \rangle|^2 - |\langle \bar{f} | T | \bar{M} \rangle|^2 = -4A_1A_2 \sin(\delta_1 - \delta_2) \sin(\phi_1 - \phi_2)$
\[ B^\pm \to \pi^\pm \pi^+ \pi^- \text{ and } B^\pm \to \pi^\pm K^+ K^- \]

\[ \Gamma(M \to f) - \Gamma(\bar{M} \to \bar{f}) = |\langle f | T | M \rangle|^2 - |\langle \bar{f} | T | \bar{M} \rangle|^2 = -4 A_1 A_2 \sin(\delta_1 - \delta_2) \sin(\phi_1 - \phi_2) \]

CPT: \[ \Gamma_{\text{total}} = \Gamma_{\text{total}} \]

\[ \Gamma_{\text{total}} = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 + \Gamma_6 + ... \]

\[ \bar{\Gamma}_{\text{total}} = \bar{\Gamma}_1 + \bar{\Gamma}_2 + \bar{\Gamma}_3 + \bar{\Gamma}_4 + \bar{\Gamma}_5 + \bar{\Gamma}_6 + ... \]

CPV in one channel should be compensated by another one with opposite sign

large CPV in low mass region related to \( \pi\pi \leftrightarrow KK \) rescattering.

Patricia Magalhães, LHCb: PRD101 (2020) no.1, 012006

BELLE: PRD 96 (2017) 3, 031101
LHCb: PRD90 (2014) 11, 112004
LHCb: PRL 124 (2020) no.3, 031801
LHCb: PRL101 (2020) no.1, 012006
$B^\pm \to \pi^\pm \pi^+ \pi^-$

Color scale:
\[
\frac{\Gamma(B^+ \to \pi^+ \pi^- \pi^+) - \Gamma(B^- \to \pi^- \pi^+ \pi^-)}{\Gamma(B^+ \to \pi^+ \pi^- \pi^+) + \Gamma(B^- \to \pi^- \pi^+ \pi^-)}
\]

$\Gamma(M \to f) - \Gamma(M \to \bar{f}) = |\langle f \mid T \mid M \rangle|^2 - |\langle \bar{f} \mid T \mid M \rangle|^2 = -4A_1 A_2 \sin(\delta_1 - \delta_2) \sin(\phi_1 - \phi_2)$

theory paper reproduces key features, but approach needs to be verified in full amplitude analysis on data.

BTW, in this context also note:

Magalhães et al - PLB 806 (2020) 135490

Magalhães et al - PLB 780 (2018) 357
Summary

- Building a model is as much an art as a science.

- Algorithmic methods for searching through the vast parameter space are very valuable.

- But they need adult supervision.

- Various methods can give additional insights - helicity distributions, moments, model-independent methods.

- An amplitude analysis that initially doesn’t quite fit can be the source of exciting new discoveries - unexpected $J^{PC}$, new particle, new insights into strong dynamics.

- Model composition can not be unlinked from the description of the components themselves. Sometimes, more sophisticated descriptions than Breit Wigners will be needed. And sometimes, an altogether different approach.
Summary

- Building a model is as much an art as a science.

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- Model composition can not be unlinked from the description of the components themselves. Sometimes, more sophisticated descriptions than Breit Wigners will be needed. And sometimes, an altogether different approach.
Lineshape of 3-body resonances

\[ a_1(1270)^+[S] \rightarrow \pi^+ (\rho(770) \rightarrow \pi^+ \pi^-) \]

\[ a_1(1270)^+[D] \rightarrow \pi^+ (\rho(770) \rightarrow \pi^+ \pi^-) \]
Figure 6.3: Magnitude-squared (a), phase (b) and Argand diagram (c) of the quasi-model-independent $a_1(1640)$ lineshape. The fitted knots are displayed as points with error bars and the black line shows the interpolated spline. The Breit-Wigner lineshape with the mass and width from the nominal fit is superimposed (red area). The latter is chosen to agree with the interpolated spline at the point $\Re(A) = 1, \Im(A) = 0$.

Figure 6.4: Magnitude-squared (a), phase (b) and Argand diagram (c) of the quasi-model-independent $a_1(1260)$ lineshape. The fitted knots are displayed as points with error bars and the black line shows the interpolated spline. The Breit-Wigner lineshape with the mass and width from the nominal fit is superimposed (red area). The latter is chosen to agree with the interpolated spline at the point $\Re(A) = 1, \Im(A) = 0$.

Figure 6.5: Magnitude-squared (a), phase (b) and Argand diagram (c) of the quasi-model-independent $\pi(1300)$ lineshape. The fitted knots are displayed as points with error bars and the black line shows the interpolated spline. The Breit-Wigner lineshape with the mass and width from the nominal fit is superimposed (red area). The latter is chosen to agree with the interpolated spline at the point $\Re(A) = 1, \Im(A) = 0$. 

Jonas Rademacker

How to Build an Amplitude Model

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$m^2(D^+ D^-) \ [\text{GeV}^2/c^4]$ vs. $m^2(D^- K^+) \ [\text{GeV}^2/c^4]$

LHCb preliminary
Helicity angles and Dalitz plot variables

\[ \cos \theta_{\text{hel}} \]

\[ m^2_{i3} \text{ (GeV/c}^2) \]

\[ m^2_{i3} \text{ (GeV/c}^2) \]

Model including $D^- K^+$ resonances

Fit results: lineshapes

<table>
<thead>
<tr>
<th>Resonance</th>
<th>Mass (GeV/c$^2$)</th>
<th>Width (MeV/c$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{c0}(3930)$</td>
<td>$3.9238 \pm 0.0015 \pm 0.0004$</td>
<td>$17.4 \pm 5.1 \pm 0.8$</td>
</tr>
<tr>
<td>$\chi_{c2}(3930)$</td>
<td>$3.9268 \pm 0.0024 \pm 0.0008$</td>
<td>$34.2 \pm 6.6 \pm 1.1$</td>
</tr>
<tr>
<td>$\chi_0(2900)$</td>
<td>$2.8663 \pm 0.0065 \pm 0.0020$</td>
<td>$57.2 \pm 12.2 \pm 4.1$</td>
</tr>
<tr>
<td>$\chi_1(2900)$</td>
<td>$2.9041 \pm 0.0048 \pm 0.0013$</td>
<td>$110.3 \pm 10.7 \pm 4.3$</td>
</tr>
</tbody>
</table>
Model including $D^- K^+$ resonances

Zoom in on the $X_{cJ}(3930)$ ($15 \text{ GeV} \cdot c^2 < m^2(D^+ D^-) < 16 \text{ GeV}/c^2$):

- **Both** a spin-0 and spin-2 component are needed in this region
- **Masses** are consistent and spin-0 slightly narrower $2002.03311$
D spectroscopy

DsJ*(2860). This structure had been seen in inclusive production prior to our Bs→DKpi Dalitz plot analysis, but without measurement of its quantum numbers.
Include Pat’s stuff

B→3h ultimate need for new (3-body) components
Pat’s building kit
Mention freed isobars
Lasso illustration (by Philippe d’Argent)

- Generated: $pdf = 1 + x$
- Fitted $pdf = 1 + \sum_{i=1}^{10} c_i x^i$
- $-2 \cdot \log(L) \rightarrow -2 \cdot \log(L) + \lambda \cdot \sum_i |c_i|$

\[\lambda = 0.0\]

- $c_1 = -0.9500 \pm 2.1580$
- $c_2 = 10.6545 \pm 7.1052$
- $c_3 = -6.9912 \pm 11.8217$
- $c_4 = -10.2646 \pm 13.3412$
- $c_5 = -3.4468 \pm 14.7392$
- $c_6 = 5.2567 \pm 15.4221$
- $c_7 = 10.3363 \pm 16.0175$
- $c_8 = 9.1735 \pm 16.8335$
- $c_9 = 0.9337 \pm 16.6720$
- $c_{10} = -14.1801 \pm 12.4028$
Lasso illustration (by Philippe d’Argent)

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- \(-2 \cdot \log(L) \rightarrow -2 \cdot \log(L) + \lambda \cdot \sum_i |c_i|\)

\(|c_1| = 2.0274 \pm 3.0131\)
\(|c_2| = -0.0012 \pm 9.7770\)
\(|c_3| = -0.8546 \pm 14.0068\)
\(|c_4| = -0.0746 \pm 16.3614\)
\(|c_5| = 0.0010 \pm 19.8780\)
\(|c_6| = -0.0005 \pm 19.8414\)
\(|c_7| = -0.0003 \pm 0.3167\)
\(|c_8| = -0.0002 \pm 0.3105\)
\(|c_9| = -0.0001 \pm 0.2857\)
\(|c_{10}| = -0.0012 \pm 5.2114\)
Lasso illustration (by Philippe d’Argent)

- Generated: pdf = 1 + x
- Fitted pdf = 1 + \( \sum_{i=1}^{10} c_i x^i \)
- \(-2 \cdot \log(L) \rightarrow -2 \cdot \log(L) + \lambda \cdot \sum_i |c_i|\)

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Lasso illustration (by Philippe d’Argent)

- **Generated:** pdf = 1 + x
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\]
Figure 13: Legendre moments up to order 7 calculated as a function of $m(K\pi^\pm)$ for data (black data points) and the fit result (solid blue curve).
https://github.com/jdalseno/Mint2
### $a_1(1260)$ WIDTH

<table>
<thead>
<tr>
<th>VALUE (MeV)</th>
<th>EVTS</th>
<th>DOCUMENT ID</th>
<th>TECN</th>
<th>COMMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>250 to 600 OUR ESTIMATE</td>
<td>367±9$^+$28$^-$25</td>
<td>ALEKSEEV 10 COMP</td>
<td>420k</td>
<td>190 $\pi^- P b \rightarrow \pi^- \pi^- \pi^+ P b'$</td>
</tr>
<tr>
<td>410±31±30</td>
<td>18 AUBERT 07AU BABR</td>
<td>10.6 $e^+ e^- \rightarrow \rho^0 \rho^+ \pi^+$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>520–680</td>
<td>6360</td>
<td>07A FOC5</td>
<td>19 LINK</td>
<td>$D^0 \rightarrow \pi^- \pi^- \pi^+$</td>
</tr>
<tr>
<td>480±20</td>
<td>19 LINK</td>
<td>07A FOC5</td>
<td>20 GOMEZ-DUM.04 RVUE</td>
<td>$\tau^+ \rightarrow \pi^+ \pi^- \pi^- \nu$</td>
</tr>
<tr>
<td>580±41</td>
<td>90k</td>
<td>04 OBLX</td>
<td>21 DRUKSYOY 02 BELL</td>
<td>$p p \rightarrow 2\pi^+ 2\pi^-$</td>
</tr>
<tr>
<td>460±85</td>
<td>205</td>
<td>04 OBLX</td>
<td>22 ASNER 00 CLE2</td>
<td>$B \rightarrow D(\pm) K^- K^0$</td>
</tr>
<tr>
<td>814±36±13</td>
<td>37k</td>
<td>00 CLE2</td>
<td>23 AKHMETSHIN 99E CMD2</td>
<td>$10.6 e^+ e^- \rightarrow \tau^+ \tau^-$</td>
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<tr>
<td>450±50</td>
<td>22k</td>
<td>00 CLE2</td>
<td>24 BONDAR 99 RVUE</td>
<td>$1.05–1.38 e^+ e^- \rightarrow \pi^+ \pi^- \pi^0 \nu$</td>
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<tr>
<td>570±10</td>
<td>98 G LPHY</td>
<td>25 ABREU 98G DLPH</td>
<td>26 ABREU 98G DLPH</td>
<td>$e^+ e^- \rightarrow 4\pi, \tau \rightarrow 3\pi \nu$</td>
</tr>
<tr>
<td>478±3±15</td>
<td>85</td>
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<td>27,28 ABREU 98G DLPH</td>
<td>$e^+ e^- \rightarrow 3\pi \nu$</td>
</tr>
<tr>
<td>425±14±8</td>
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<tr>
<td>621±32±58</td>
<td>25,29 ACKERSTAFF 97R OPAL</td>
<td>$E_{C_{M}} = 88–94, \tau \rightarrow 3\pi \nu$</td>
<td></td>
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<tr>
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</tbody>
</table>
Strong phase measured in 5 phase-space bins

\[ c_i^{4\pi} = \int_{\text{bin } i} |A_D||A_{\bar{D}}|\cos(\delta_D) \propto \frac{N_{CP+}}{N_{CP+} + N_{CP-}} \]

\[ s_i^{4\pi} = \int_{\text{bin } i} |A_D||A_{\bar{D}}|\sin(\delta_D) \]

**Binning optimized for sensitivity to \( \gamma \) (gain of factor 2.2)**

**Good consistency with model prediction:**

\[ \chi^2/ndof = 13.7/10 \quad (p = 0.19) \]

**Estimated sensitivity** \( \sigma(\gamma) \approx 12^\circ (5^\circ) \) for LHCb-Run-II(III)
Maybe mention staged Lasso in KKpipi
\[ \text{B}_s, \text{B}_s \rightarrow \text{D}^+_s \text{K}^+ \pi^- \pi^+ \text{ time-dependent amplitude analysis} \]

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
(a) Cascade decays & (b) Quasi-two-body decays \\
\hline
\( B_s^0 \rightarrow D_s^+ (K_1(1270)^\pm \rightarrow \pi^\pm K^*(892)^0) \) & \( B_s^0 \rightarrow f_0(500)^0 (D_s^\pm K^\mp)^0 \) \\
\( B_s^0 \rightarrow D_s^+ (K_1(1270)^\pm \rightarrow \pi^\pm K_2^*(1430)^0) \) & \( B_s^0 \rightarrow f_0(500)^0 (D_s^\pm K^\mp)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K_1(1270)^\pm \rightarrow K^+ f_0(500)^0] \) & \( B_s^0 \rightarrow f_0(980)^0 (D_s^\pm K^\mp)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K_1(1270)^\pm \rightarrow K^+ f_0(980)^0] \) & \( B_s^0 \rightarrow \rho(770)^0 (D_s^\pm K^\mp)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K_1(1270)^\pm \rightarrow K^+ \rho(770)^0] \) & \( B_s^0[S, P, D] \rightarrow \rho(770)^0 (D_s^\pm K^\mp)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K_1(1270)^\pm \rightarrow K^+ \rho(1450)^0] \) & \( B_s^0 \rightarrow K^*_0(1430)^0 (D_s^\pm \pi^\pm)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K_1(1400)^0 \rightarrow \pi^\pm K^*(892)^0] \) & \( B_s^0 \rightarrow K^*_0(1430)^0 (D_s^\pm \pi^\pm)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K_1(1400)^0 \rightarrow \pi^\pm K^*(892)^0] \) & \( B_s^0 \rightarrow K^*_2(1430)^0 (D_s^\pm \pi^\pm)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K(1460)^0 \rightarrow K^+ \rho(770)^0] \) & \( B_s^0 \rightarrow (D_s^\pm \pi^\pm)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K(1460)^0 \rightarrow \pi^\pm K^*(892)^0] \) & \( B_s^0 \rightarrow K^*_2(1430)^0 (\pi^\pm \pi^\pm)^0 \) \\
\( B_s^0 \rightarrow D_s^+ [K(1460)^0 \rightarrow K^* \rho(770)^0] \) & \( B_s^0 \rightarrow (D_s^\pm \pi^\pm)^0 \) \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|l|l|l|}
\hline
Decay channel & \( F_i^c \) & \( F_i^w \) \\
\hline
\( B_s^0 \rightarrow D_s^+ (K_1(1270)^\pm \rightarrow K^*(892)^0 \pi^\pm) \) & 13.0 \pm 2.4 \pm 2.7 \pm 3.4 & 4.1 \pm 2.2 \pm 2.9 \pm 2.6 \\
\( B_s^0 \rightarrow D_s^+ (K_1(1270)^\pm \rightarrow K^+ \rho(770)^0) \) & 16.0 \pm 1.4 \pm 1.8 \pm 2.1 & 5.1 \pm 2.2 \pm 3.5 \pm 2.0 \\
\( B_s^0 \rightarrow D_s^+ (K_1(1270)^\pm \rightarrow K^+ \rho(1450)^0) \) & 3.4 \pm 0.5 \pm 1.0 \pm 0.4 & 1.1 \pm 0.5 \pm 0.6 \pm 0.5 \\
\( B_s^0 \rightarrow D_s^+ (K_1(1400)^0 \rightarrow K^* \rho(770)^0) \) & 63.9 \pm 5.1 \pm 7.4 \pm 13.5 & 19.3 \pm 5.2 \pm 8.3 \pm 7.8 \\
\( B_s^0 \rightarrow D_s^+ (K_1(1400)^0 \rightarrow K^* \rho(1450)^0) \) & 12.8 \pm 0.8 \pm 1.5 \pm 3.2 & 12.6 \pm 2.0 \pm 2.6 \pm 4.1 \\
\( B_s^0 \rightarrow D_s^+ (K^*(1410)^0 \rightarrow K^* \rho(770)^0) \) & 5.6 \pm 0.4 \pm 0.6 \pm 0.7 & 5.6 \pm 1.0 \pm 1.2 \pm 1.8 \\
\( B_s^0 \rightarrow D_s^+ (K^*(1410)^0 \rightarrow K^* \rho(1450)^0) \) & 11.9 \pm 2.5 \pm 2.9 \pm 3.1 & 11.9 \pm 2.5 \pm 2.9 \pm 3.1 \\
\( B_s^0 \rightarrow (D_s^\pm \pi^\pm)^0 K^*(892)^0 \) & 10.2 \pm 1.6 \pm 1.8 \pm 4.5 & 28.4 \pm 5.6 \pm 6.4 \pm 15.3 \\
\( B_s^0 \rightarrow (D_s^\pm \pi^\pm)^0 K^*(892)^0 \) & 0.9 \pm 0.4 \pm 0.5 \pm 1.0 & 0.9 \pm 0.4 \pm 0.5 \pm 1.0 \\
\hline
Sum & 125.7 \pm 6.4 \pm 6.9 \pm 19.9 & 88.1 \pm 7.0 \pm 10.0 \pm 20.9 \\
\hline
\end{tabular}
\end{table}
**B_{s}, B_{s} \rightarrow D_{s}^{-} K^{+} \pi^{-} \pi^{+}** time-dependent amplitude analysis

**B_s, B_s → D_s^- K^+ π^- π^+ time-dependent amplitude analysis**

The systematic uncertainties on the measured observables are summarised in Table 4 for between the fitted and generated values by the statistical uncertainty. The means of the each pseudoexperiment and fit parameter, a pull is calculated by dividing the di

Others

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Model-independent</th>
<th>Model-dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>0.47 ± 0.08 ± 0.02</td>
<td>0.56 ± 0.05 ± 0.04 ± 0.07</td>
</tr>
<tr>
<td>κ</td>
<td>0.88 ± 0.12 ± 0.04</td>
<td>0.72 ± 0.04 ± 0.06 ± 0.04</td>
</tr>
<tr>
<td>δ [°]</td>
<td>-6 ± 10 ± 2</td>
<td>-14 ± 10 ± 4 ± 5</td>
</tr>
<tr>
<td>γ - 2β_s [°]</td>
<td>42 ± 19 ± 6</td>
<td>42 ± 10 ± 4 ± 5</td>
</tr>
</tbody>
</table>