OPE in Belle II Analyses

Part 1: Basics of Quantum Field Theory

Tatsu Takeuchi

Virginia Tech

Belle II Summer Workshop, July 13, 2021

What is an Operator Product Expansion (OPE) ?

An Operator Product Expansion (OPE) is defined as an expansion of a product of operator-valued (quantum) fields evaluated at two closely placed spacetime points in terms of "local" operators :

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \;=\; \sum_i C_i(x)\mathcal{O}_i(0)\;.$$

The expansion coefficients $C_i(x)$ are called the Wilson coefficients. (Named after Kenneth G. Wilson (1936-2013), Nobel Prize in Physics 1982.)

The behavior of the operator product as x is varied is encapsulated in the Wilson coefficients.

Why is this relevant for Belle II?

What is an Operator Product Expansion (OPE) ?

Many OPE-type expressions appear in the **Belle II Physics Book** (PTEP 2019, 123C01).

Examples:

1. Eq. (62), the effective interaction Hamiltonian for $b \to c\bar{u}d$:

$$H^{b\to c\bar{u}d} = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \sum_{j=1,2} C_j\, Q_j^{c\bar{u}d} ,$$

where

$$Q_1^{c\bar{u}d} = (\overline{d}_L^{\alpha}\gamma_\mu u_L^{\beta})(\overline{c}_L^{\beta}\gamma^\mu b_L^{\alpha}) , \qquad Q_2^{c\bar{u}d} = (\overline{d}_L^{\alpha}\gamma_\mu u_L^{\alpha})(\overline{c}_L^{\beta}\gamma^\mu b_L^{\beta}) .$$

Here, α and β are color indices. See also Eq. (344).

What is an Operator Product Expansion (OPE) ?

2. Eq. (136), the effective interaction Hamiltonian for possible new physics effects in $b \to c\tau\nu$:

$$
\begin{aligned}
-\mathcal{L}_{\text{eff}} &= 2\sqrt{2}G_F V_{cb}\Big[(1 + C_{V_1})\mathcal{O}_{V_1} + C_{V_2}\mathcal{O}_{V_2} \\
&\qquad + C_{S_1}\mathcal{O}_{S_1} + C_{S_2}\mathcal{O}_{S_2} + C_T\mathcal{O}_T\Big]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{O}_{V_1} &= (\overline{c}_L\gamma_\mu b_L)(\overline{\tau}_L\gamma^\mu\nu_L)\,, & \mathcal{O}_{V_2} &= (\overline{c}_R\gamma_\mu b_R)(\overline{\tau}_L\gamma^\mu\nu_L)\,, \\
\mathcal{O}_{S_1} &= (\overline{c}_L b_R)(\overline{\tau}_R\nu_L)\,, & \mathcal{O}_{S_2} &= (\overline{c}_R b_L)(\overline{\tau}_R\nu_L)\,, \\
\mathcal{O}_T &= (\overline{c}_R\sigma_{\mu\nu} b_L)(\overline{\tau}_R\sigma^{\mu\nu}\nu_L)\,.
\end{aligned}
$$

(Note that $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$ in the absence of derivatives in $\mathcal{L}_{\text{int}}$.)

3. Eq. (147) gives a similar expression for $b \to u\tau\nu$, where c is replaced by u everywhere.

VIRGINIA TECH

What is an Operator Product Expansion (OPE) ?

4. Eq. (417), the effective interaction Hamiltonian relevant for possible new physics effects in $c \to u\ell^+\ell^-$ ($D^0 \to \ell^+\ell^-$) :

$$\mathcal{H}_{\mathrm{NP}}^{\mathrm{rare}} = \sum_{i=1}^{10} \frac{\tilde{C}_i(\mu)}{\Lambda^2} \tilde{\mathcal{Q}}_i \,,$$

where

$$\begin{aligned} \tilde{\mathcal{Q}}_1 &= (\bar{\ell}_L \gamma_\mu \ell_L)(\overline{u}_L \gamma^\mu c_L)\,, & \tilde{\mathcal{Q}}_2 &= (\bar{\ell}_L \gamma_\mu \ell_L)(\overline{u}_R \gamma^\mu c_R)\,, \\ \tilde{\mathcal{Q}}_3 &= (\bar{\ell}_L \ell_R)(\overline{u}_R c_L)\,, & \tilde{\mathcal{Q}}_4 &= (\bar{\ell}_R \ell_L)(\overline{u}_R c_L)\,, \\ \tilde{\mathcal{Q}}_5 &= (\bar{\ell}_R \sigma_{\mu\nu} \ell_L)(\overline{u}_R \sigma^{\mu\nu} c_L)\,. \end{aligned}$$

$\tilde{\mathcal{Q}}_{6\sim10}$ are obtained from $\tilde{\mathcal{Q}}_{1\sim5}$ via the interchange $L \leftrightarrow R$.

What is an Operator Product Expansion (OPE) ?

- Wait!
  - The interaction Hamiltonian $\mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int}$ on the left-hand sides are not products of operators evaluated at two different spacetime points, or are they!?
  - Where are the x-dependences of the Wilson coefficients? Is the μ in the Wilson coefficients $\tilde{C}_i(\mu)$ of Eq. (417) the same thing as x?
  - What is the Fermi constant G_F doing in Eqs. (62), (136), and (147)? And what is Λ in Eq. (417)?
  - etc.
- In order to understand what is going on, we need to introduce the concepts of "renormalization scale μ," and "effective QFT at μ"
- We will begin by reviewing the basics of Quantum Field Theory (QFT) so that we know what we mean when we say "operator".

Classical Harmonic Oscillator - Lagrangian

Consider the classical Lagrangian for a harmonic oscillator:

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \,.$$

The equation of motion and generic solution are

$$\begin{aligned} 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} &= m\ddot{q} + kq \\ &\downarrow \\ q(t) &= A\,e^{-i\omega t} + A^*e^{i\omega t}\,, \qquad \omega = \sqrt{\frac{k}{m}}\,. \end{aligned}$$

Classical Harmonic Oscillator - Hamiltonian

Momentum conjugate to q is

$$p \;=\; \frac{\partial L}{\partial \dot{q}} \;=\; m\dot{q}\;,$$

and the Hamiltonian is

$$H \;=\; p\dot{q} - L \;=\; \frac{p^2}{2m} + \frac{1}{2}kq^2\;.$$

The equations of motion and solution are

$$\begin{aligned} \dot{q} &= \{q, H\} \;=\; \frac{p}{m}\;, \\ \dot{p} &= \{p, H\} \;=\; -kq\;, \\ &\downarrow \\ q(t) &= A\,e^{-i\omega t} + A^* e^{i\omega t}\;, \qquad \omega \;=\; \sqrt{\frac{k}{m}}\;. \end{aligned}$$

Quantum Harmonic Oscillator

Quantization (Heisenberg picture) :

$$q \;\rightarrow\; \hat{q}\;, \qquad p \;\rightarrow\; \hat{p}\;, \qquad \{q,p\} \;=\; 1 \;\rightarrow\; \frac{1}{i\hbar}[\hat{q},\hat{p}] \;=\; 1\;.$$

The equations of motion and solution are

$$\begin{aligned}
\dot{\hat{q}} &= \frac{1}{i\hbar}[\hat{q},\hat{H}] \;=\; \frac{\hat{p}}{m}\;,\\
\dot{\hat{p}} &= \frac{1}{i\hbar}[\hat{p},\hat{H}] \;=\; -k\hat{q}\;,\\
&\downarrow\\
\hat{q}(t) &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}\,e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\;, \qquad \omega \;=\; \sqrt{\frac{k}{m}}\;,\\
\frac{1}{i\hbar}[\hat{q},\hat{p}] &= 1 \quad\rightarrow\quad [\hat{a},\hat{a}^\dagger] \;=\; 1\;.
\end{aligned}$$

Quantum Harmonic Oscillator

$\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators:

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle\;, \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\;, \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\;,$$

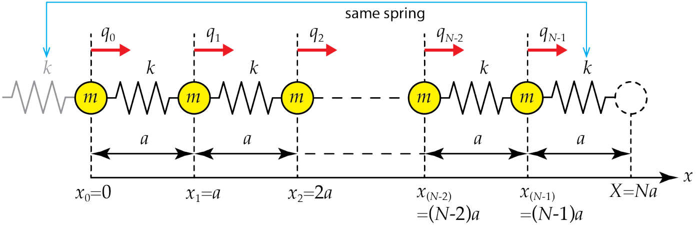
where $n = 0, 1, 2, \cdots$.

Hamiltonian and energy eigenvalues:

$$\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{q}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \\
&\downarrow \\
E_n &= \hbar\omega\left(n + \frac{1}{2}\right)\;.
\end{aligned}$$

1D Lattice Oscillations - Lagrangian

Consider N equal masses connected by N equal springs with periodic boundary condition ($q_N = q_0$) :



The Lagrangian and equation of motion are

$$
\begin{aligned}
L &= \frac{1}{2}m\sum_{j=0}^{N-1}\dot{q}_j^{\,2} - \frac{1}{2}k\sum_{j=0}^{N-1}(q_{j+1}-q_j)^2\;, \\
0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = m\ddot{q}_j + k\Big[-q_{j-1}+2q_j-q_{j+1}\Big]
\end{aligned}
$$

1D Lattice Oscillations - Lagrangian

The generic solution is

$$q_j(t) \;=\; \frac{1}{\sqrt{N}}\sum_n \left\{ A_n e^{-i(\omega_n t - j\theta_n)} + A_n^* e^{i(\omega_n t - j\theta_n)} \right\}$$

where

$$\theta_n \;=\; \frac{2n\pi}{N}\;, \qquad \omega_n \;=\; 2\sqrt{\frac{k}{m}}\left|\sin\frac{\theta_n}{2}\right|\;, \qquad n\in\mathbb{Z}\;.$$

If we label q_j with its equilibrium position $x_j = ja$ along the x-axis instead of the integer j, then we can write

$$q(t,x_j) \;=\; \frac{1}{\sqrt{N}}\sum_n \left\{ A_n e^{-i(\omega_n t - k_n x_j)} + A_n^* e^{i(\omega_n t - k_n x_j)} \right\}$$

where

$$k_n \;=\; \frac{\theta_n}{a} \;=\; \frac{2n\pi}{Na}$$

is the wave-number.

1D Lattice Oscillations - Hamiltonian

The momentum conjugate to q_j is

$$p_j \;=\; \frac{\partial L}{\partial \dot{q}_j} \;=\; m\dot{q}_j \;,$$

and the Hamiltonian is

$$H \;=\; \sum_j p_j \dot{q}_j - L \;=\; \frac{1}{2m}\sum_{j=0}^{N-1} p_j^2 + \frac{1}{2}k\sum_{j=0}^{N-1}(q_{j+1}-q_j)^2 \;.$$

The equations of motion are

$$\begin{aligned}
\dot{q}_j \;&=\; \{q_j, H\} \;=\; \frac{p_j}{m}\;,\\
\dot{p}_j \;&=\; \{p_j, H\} \;=\; k(q_{j+1} - 2q_j + q_{j-1})\;,
\end{aligned}$$

which are equivalent to the Euler-Lagrange equations.

1D Lattice Oscillations - Quantization

To quantize, we make the replacements

$$q_j \;\rightarrow\; \hat{q}_j \qquad p_j \;\rightarrow\; \hat{p}_j \qquad \{q_i, p_j\} \;=\; \delta_{ij} \;\rightarrow\; \frac{1}{i\hbar}[\hat{q}_i, \hat{p}_j] \;=\; \delta_{ij}$$

The equations of motion are

$$\begin{aligned}
\dot{\hat{q}}_j &= \frac{1}{i\hbar}[\hat{q}_j, \hat{H}] \;=\; \frac{\hat{p}_j}{m}\;, \\
\dot{\hat{p}}_j &= \frac{1}{i\hbar}[\hat{p}_j, \hat{H}] \;=\; k\left(\hat{q}_{j+1} - 2\hat{q}_j + \hat{q}_{j-1}\right)\;,
\end{aligned}$$

which are exactly the same as the classical equations.
The solution is

$$\hat{q}_j(t) \;=\; \frac{1}{\sqrt{N}} \sum_n \sqrt{\frac{\hbar}{2m\omega_n}} \left\{ \hat{a}_n e^{-i(\omega_n t - j\theta_n)} + \hat{a}_n^\dagger e^{i(\omega_n t - j\theta_n)} \right\}$$

1D Lattice Oscillations - Quantization

or

$$\hat{q}(t, x_j) = \frac{1}{\sqrt{N}} \sum_n \sqrt{\frac{\hbar}{2m\omega_n}} \left\{ \hat{a}_n e^{-i(\omega_n t - k_n x_j)} + \hat{a}_n^\dagger e^{i(\omega_n t - k_n x_j)} \right\}$$

where

$$\frac{1}{i\hbar}[\hat{q}_i, \hat{p}_j] = \delta_{ij} \quad \rightarrow \quad [\hat{a}_m, \hat{a}_n^\dagger] = \delta_{mn} ,$$

with all other commutators zero.

The Hamiltonian in terms of the creation and annihilation operators are

$$\hat{H} = \frac{1}{2m} \sum_{j=0}^{N-1} \hat{p}_j^2 + \frac{1}{2} k \sum_{j=0}^{N-1} (\hat{q}_{j+1} - \hat{q}_j)^2 = \sum_n \hbar\omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right) .$$

1D Lattice Oscillations - Quantization

If we define

$$\hat{P} \;=\; \sum_n \hbar k_n \, \hat{a}_n^\dagger \hat{a}_n \;,$$

then $\hat{P}$ is the generator of translations in x-space since it is straightforward to show that the operator

$$\hat{T} \;=\; e^{i\hat{P}a}$$

transforms q_j to q_{j+1} :

$$\hat{T}\hat{q}_j\hat{T}^{-1} \;=\; \hat{q}_{j+1} \;, \quad \text{or} \quad \hat{T}\hat{q}(x_j)\hat{T}^{-1} \;=\; \hat{q}(x_{j+1}) \;=\; \hat{q}(x_j + a) \;.$$

So we can identify $\hat{P}$ with the momentum operator.
So the operator $\hat{a}_n^\dagger$ and $\hat{a}_n$ respectively increases and decreases the energy and momentum of the system in units of $\hbar\omega_n$ and $\hbar k_n$.

We can use quantized fields to model particles!

Particle Interpretation

At each lattice site, there exists a field operator given by

$$\hat{q}(t, x_j) \;=\; \frac{1}{\sqrt{N}}\sum_n \sqrt{\frac{\hbar}{2m\omega_n}}\left\{\hat{a}_n e^{-i(\omega_n t - k_n x_j)} + \hat{a}_n^\dagger e^{i(\omega_n t - k_n x_j)}\right\}$$

which can either create or annihilate a particle. This operator can also be thought of as residing in a cell of width a centered at x_j.

The potential energy includes terms that annihilates a particle at some point x_j on the lattice and then recreates it at a neighboring point $x_{j\pm1}$, describing propagation of the particle from one cell to the next.

For finite lattice spacing a, the momentum of the particle is limited to the 1st Brillouin Zone:

$$-\frac{\pi}{a} \;<\; k_n \;<\; \frac{\pi}{a}\;.$$

For particles with momenta well inside this zone, the lattice spacing a will not be noticable. (Wavelengths are much longer than a.)

Taking the Continuum Limit

We usually take the continuum ($a \to 0$) and infinite volume ($Na \to \infty$) limits for the ease of imposing Lorentz covariance (at the expense of introducing various infinities). The $\hat{q}_j$ and $\hat{p}_j$ operators are rescaled

$$\frac{\hat{q}(x_j)}{\sqrt{a}} \;\to\; \hat{\phi}(t,x)\;, \qquad \frac{\hat{p}(x_j)}{\sqrt{a}} \;\to\; \hat{\pi}(t,x)\;,$$

so that

$$[\hat{q}(t,x_i),\hat{p}(t,x_j)] \;=\; i\hbar\delta_{ij} \quad\to\quad [\hat{\phi}(t,x),\hat{\pi}(t,y)] \;=\; i\hbar\delta(x-y)$$

(for Bosons). Sums over $x_j = ja$ and k_n are replaced by integrals:

$$\sum_{j=0}^{N-1} f(x_j) \;\to\; \frac{1}{a}\int_0^{Na} dx\, f(x)\;, \qquad \sum_{n=-[N/2]}^{[N/2]} g(k_n) \;\to\; Na\int_{-\pi/a}^{\pi/a}\frac{dk}{2\pi}\,g(k)\;.$$

Fields used in Particle Physics – Real Scalar

We work in $3+1$ dimensions and use natural units in which $\hbar = c = 1$. We also omit hats on the operators.

- Real Scalar Field – models chargeless spin-0 particle for which antiparticle = particle.

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \;, \\
0 &= (\partial^2 + m^2)\phi \;, \\
\phi(x) &= \int \frac{d^3\mathrm{k}}{(2\pi)^3 2\omega_{\mathrm{k}}} \left(a(\mathrm{k})\, e^{-ikx} + a^\dagger(\mathrm{k})\, e^{ikx} \right) \;, \\
[a(\mathrm{k}), a^\dagger(\mathrm{k}')] &= (2\pi)^3 2\omega_{\mathrm{k}} \delta^3(\mathrm{k}-\mathrm{k}') \;,
\end{aligned}$$

where $\omega_{\mathrm{k}} = \sqrt{\mathrm{k}^2 + m^2}$. The normalization used here is relativistic.

Fields used in Particle Physics – Complex Scalar

- Complex Scalar Field – models spin-0 particle with charge (antiparticle $\neq$ particle).

$$
\begin{aligned}
\mathcal{L} &= \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \,, \\
0 &= (\partial^2 + m^2)\phi \\
\phi(x) &= \int \frac{d^3\mathrm{k}}{(2\pi)^3 2\omega_{\mathrm{k}}} \left(a(\mathrm{k})\, e^{-ikx} + c^\dagger(\mathrm{k})\, e^{ikx} \right) \,, \\
[a(\mathrm{k}), a^\dagger(\mathrm{k}')] &= [c(\mathrm{k}), c^\dagger(\mathrm{k}')] \;=\; (2\pi)^3 2\omega_{\mathrm{k}} \delta^3(\mathrm{k} - \mathrm{k}') \,.
\end{aligned}
$$

The states created by $a^\dagger$ and those created by $c^\dagger$ have opposite charge.

VIRGINIA TECH

Fields used in Particle Physics – Dirac Spinor

- Dirac Spinor Field – models spin-$\frac{1}{2}$ particle with charge (antiparticle $\neq$ particle).

$$\begin{aligned}\mathcal{L} &= \frac{i}{2}\left\{\overline{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma^\mu\psi\right\} - m\overline{\psi}\psi \\ &= \overline{\psi}(i\partial\!\!\!/ - m)\psi + (\text{total derivative})\;,\end{aligned}$$

$$0 = (i\partial\!\!\!/ - m)\psi\;,$$

$$\psi(x) = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2\omega_{\mathrm{k}}} \sum_s \left[b(\mathrm{k},s)\,u(\mathrm{k},s)\,e^{-ikx} + d^\dagger(\mathrm{k},s)\,v(\mathrm{k},s)\,e^{ikx} \right]$$

The creation and annihilation operators satisfy anti-commutation relations :

$$\{b(\mathrm{k},s), b^\dagger(\mathrm{k}',s')\} = \{d(\mathrm{k},s), d^\dagger(\mathrm{k}',s')\} = (2\pi)^3 2\omega_{\mathrm{k}}\delta^3(\mathrm{k}-\mathrm{k}')\delta_{ss'}\;.$$

The states created by $b^\dagger$ and those created by $d^\dagger$ have opposite charge.

Fields used in Particle Physics – Majorana Spinor

- Majorana Spinor Field – models chargeless spin-$\frac{1}{2}$ particle for which antiparticle = particle. We impose the condition

$$\psi_M^C \;=\; \psi_M$$

where C denotes charge conjugation. (Subscript M is for Majorana.) The Lagrangian density and the equation of motion are

$$\mathcal{L} \;=\; \frac{1}{2}\,\overline{\psi}_M(i\partial\!\!\!/ - m)\psi_M\;, \qquad (i\partial\!\!\!/ - m)\psi_M \;=\; 0\;.$$

Solution is

$$\psi_M(x) = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2\omega_{\mathrm{k}}} \sum_s \left[b(\mathrm{k},s)\,u(\mathrm{k},s)\,e^{-ikx} + b^\dagger(\mathrm{k},s)\,v(\mathrm{k},s)\,e^{ikx} \right]$$

where

$$\{b(\mathrm{k},s), b^\dagger(\mathrm{k}',s')\} \;=\; (2\pi)^3 2\omega_{\mathrm{k}}\delta^3(\mathrm{k}-\mathrm{k}')\delta_{ss'}\;.$$

Fields used in Particle Physics – Vector

- The Lagrangian density for the massless case is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \underbrace{-\frac{\xi}{2}(\partial^\mu A_\mu)^2}_{\text{gauge fixing term}}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; .$$

The equation of motion is simplest in Feynman gauge, $\xi = 1$:

$$\partial^2 A_\mu = 0 \; .$$

The solution is

$$A^\mu = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2\omega_{\mathrm{k}}} \sum_{\lambda=0}^{3} \left[a(\mathrm{k},\lambda)\,\varepsilon^\mu(\mathrm{k},\lambda)\,e^{-ikx} + a^\dagger(\mathrm{k},\lambda)\,\varepsilon^{\mu*}(\mathrm{k},\lambda)\,e^{ikx} \right]$$

where $\lambda = 0, 1, 2, 3$ is the polarization label, and

$$[a(\mathrm{k},\lambda), a^\dagger(\mathrm{k}',\lambda')] = -g_{\lambda\lambda'}(2\pi)^3 2\omega_{\mathrm{k}}\delta^3(\mathrm{k}-\mathrm{k}') \; .$$

Interactions

- Without interactions, nothing happens!
- An interaction that occurs at spacetime point x is described by field operators multiplied together at x, e.g.

$$\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)$$

Each field is a sum of a creation operator part and an annihilation operator part:

$$\psi \sim b + d^\dagger\;, \quad \overline{\psi} \sim b^\dagger + d\;, \quad A_\mu \sim a + a^\dagger$$

If ψ describes the electron and A_μ the photon, then the above interaction can annihilate an electron with ψ, create an electron with $\overline{\psi}$, and either annihilate or create a photon with A_μ. So it can describe either photon absorption or emission by an electron.

Standard Model Interactions - Weak Interactions

See section 7, Theory Overview, of the **Belle II Physics Book**.
Eq. (49) gives the interaction of the quarks with the W-boson:

$$\mathcal{L}_W^q = \frac{g}{\sqrt{2}}\left[V_{jk}\overline{u}_{Lj}\gamma^\mu d_{Lk}W_\mu^+ + V_{jk}^*\overline{d}_{Lk}\gamma^\mu u_{Lj}W_\mu^- \right]$$

- Repeated indices (flavor indices j, k, and the Lorenz index μ) are summed (Einstein convention).
- V_{jk} are the elements of the Cabbibo-Kobayashi-Maskawa (CKM) matrix that Prof. Schwartz discussed yesterday. (See also section 7.2)
- The subscript L means that the fermion field is left-handed:

$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi\,, \qquad \psi_R = \frac{1}{2}(1+\gamma_5)\psi\,.$$

- $(u_1, u_2, u_3) = (u, c, t)$, $(d_1, d_2, d_3) = (d, s, b)$.
- Interactions of the quarks with the Z-boson are not shown.

Standard Model Interactions - Strong Interactions

The coupling of quark q to the gluons G^a_μ $(a = 1, 2, \cdots, 8)$:

$$\mathcal{L}_{\mathrm{QCD}} \;=\; g_s\, \overline{q}^{\alpha}\gamma^{\mu}(T_a)_{\alpha\beta}q^{\beta}\, G^a_\mu$$

- $\alpha, \beta = R, G, B$ are color indices.
- $T_a = \lambda_a/2$ where

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix},$$

$$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$