OPE in Belle II Analyses

Part 2: Operator Product Expansion and Wilson Coefficients

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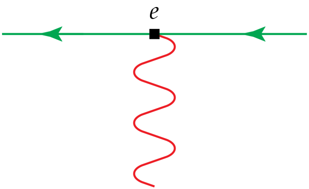
Summary of what we discussed yesterday:

- Particles are described by excitations of the corresponding quantum field.
- The quantum field at each spacetime point are operators that either create or annihilate the corresponding particle.
- Interactions between particles that happen at spacetime point x are described by products of the field operators at x. Example:

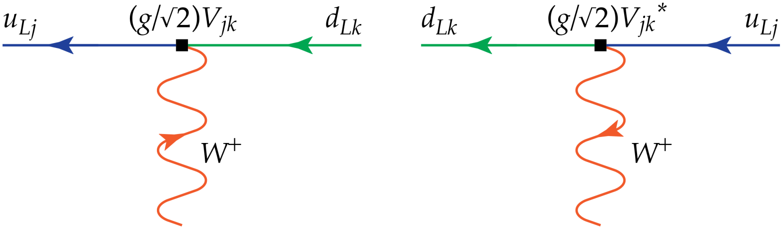
$$e\,\overline{\psi}(x)\gamma^{\mu}\psi(x)\,A_{\mu}(x)$$



Weak Interactions:

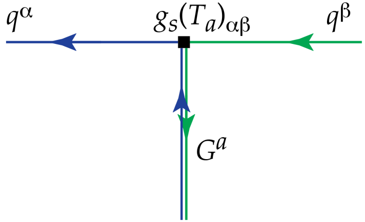
Eq. (49) of the **Belle II Physics Book**:

$$\mathcal{L}_W^q = \frac{g}{\sqrt{2}}\left[V_{jk}\overline{u}_{Lj}\gamma^\mu d_{Lk}W_\mu^+ + V_{jk}^*\overline{d}_{Lk}\gamma^\mu u_{Lj}W_\mu^-\right]$$



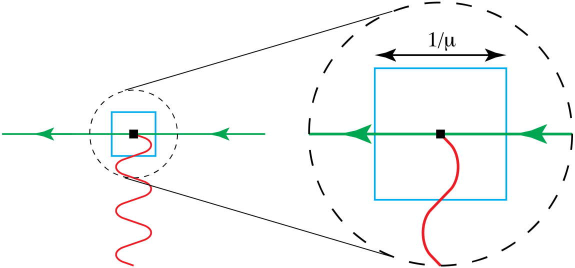
Strong Interactions:

$$\mathcal{L}_{\mathrm{QCD}} = g_s\, \overline{q}^{\alpha} \gamma^{\mu} (T_a)_{\alpha\beta} q^{\beta}\, G^a_{\mu}$$



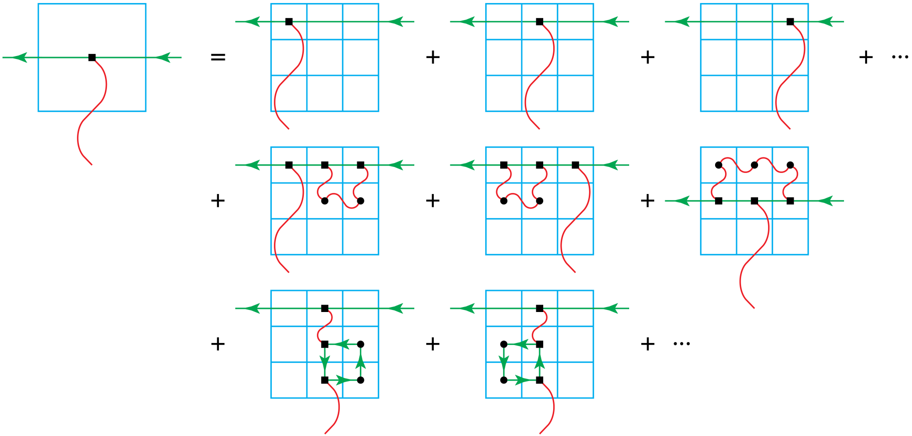
The gluon brings in color α and takes out color β.

Dependence of QFT on scale – Renormalization



- Consider the interaction $e(\overline{\psi}\gamma^{\mu}\psi)A_{\mu}$ between charged fermion ψ and the photon A_{μ}.
- If the momentum of the photon is q, its wavelength is $\lambda = 2\pi/q$.
- The photon cannot resolve spacetime points that are separated by spatial distances smaller than $\sim\lambda$.
- The photon will interact coherently at all the spacetime points inside a fuzzy cell of size $\sim\lambda$ around x.

Dependence of QFT on scale – Renormalization



- Many microscopic processes could happen inside this fuzzy cell that we cannot resolve.

Dependence of QFT on scale – Renormalization

- The effective coupling between the fermion ψ and the photon A_μ will depend on the relevant scale μ, called the renormalization scale, at which the interaction takes place. In the case, it is set by the momentum of the photon q: $\mu \sim q$.
- The interaction vertex becomes "fuzzed out" around x:

$$e(\overline{\psi}(x)\gamma^\mu\psi(x))A_\mu(x) \quad \rightarrow \quad e(\overline{\psi}(x_1)\gamma^\mu\psi(x_2))A_\mu(x_3)$$

where x_1, x_2, and x_3 are spacetime points inside the cell of size $\sim 1/\mu$ around x. In other words, the interaction Hamiltonian becomes an operator product.

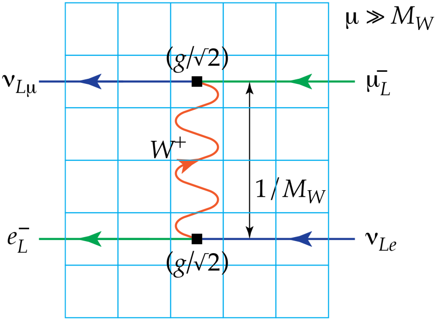
- The operator product can be made "local" again by absorbing the effects of all the microscopic processes into a running coupling constant:

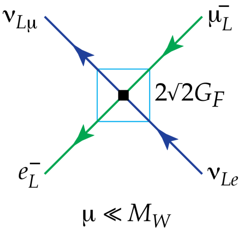
$$e(\overline{\psi}(x_1)\gamma^\mu\psi(x_2))A_\mu(x_3) \rightarrow e(q^2)\,(\overline{\psi}(x)\gamma^\mu\psi(x))A_\mu(x)$$

- An OPE with only one term!

Dependence of QFT on scale – Weak Interaction

- Exchange of W between μ_L^- and ν_{Le} :





- Range of the W-exchange force is $\sim 1/M_W$.
  (Recall how Yukawa predicted the pion mass.)
- Description depends on the renormalization scale μ.

Dependence of QFT on scale – Weak Interaction

- In the SM the process is described (at scale μ) by

$$\frac{g^2(\mu)}{2}\left[\overline{e}_L(x_2)\gamma_\lambda\nu_{Le}(x_2)\right]\underbrace{\left[-\int^{\mu}\frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip(x_2-x_1)}}{p^2-M_W^2+i\epsilon}\right]}_{W\ \text{propagator}}\left[\overline{\nu}_{L\mu}(x_1)\gamma^\lambda\mu_L(x_1)\right]\,.$$

- When $\mu \le M_W$, we can describe the process in an "effective" QFT without the W in it, and instead introduce the "local" operator

$$2\sqrt{2}G_F(\mu)\left[\overline{e}_L(x)\gamma_\lambda\nu_{Le}(x)\right]\left[\overline{\nu}_{L\mu}(x)\gamma^\lambda\mu_L(x)\right]\,.$$

- Demand that the two descriptions match at $\mu = M_W$, that is, the "effective" four-fermion term is the first term in the OPE of the SM expression:

$$\frac{g^2(M_W)}{2M_W^2} \;=\; 2\sqrt{2}G_F(M_W) \quad\rightarrow\quad G_F(M_W) \;=\; \frac{1}{\sqrt{2}v^2}\,,$$

where v is the Higgs VEV and $M_W = gv/2$.

Some technical detail :

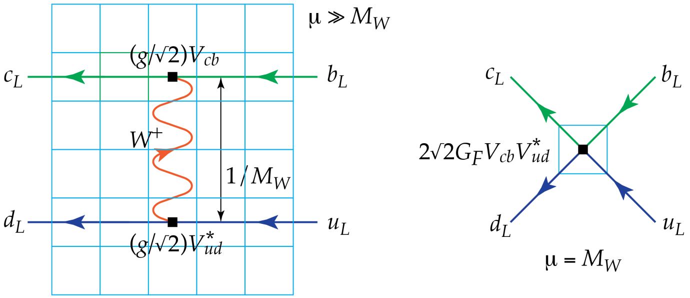
- When $\mu \leq M_W$, we can expand

$$
\begin{aligned}
& -\int^{\mu} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x_2-x_1)}}{p^2 - M_W^2 + i\epsilon} \\
&= \frac{1}{M_W^2} \int^{\mu} \frac{d^4p}{(2\pi)^4} \left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \cdots \right) e^{-ip(x_2-x_1)} \\
&= \frac{1}{M_W^2} \underbrace{\int^{\mu} \frac{d^4p}{(2\pi)^4} e^{-ip(x_2-x_1)}}_{\sim \delta^4(x_2-x_1)} + \cdots \\
&= \frac{1}{M_W^2} \delta^4_{\mu}(x_2 - x_1) + \cdots
\end{aligned}
$$

  The $\cdots$ lead to higher dimensional operators suppressed by powers of M_W^2 in the OPE.

- μ-dependence of $G_F(\mu)$ is weak.

Effect of Strong Interactions



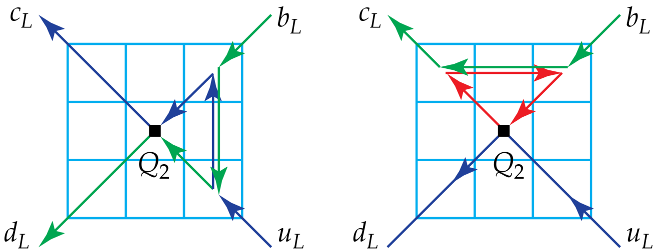
- At $\mu = M_W$, the weak interaction responsible for $b \to c\bar{u}d$ is described by the operator

$$2\sqrt{2}G_F V_{cb} V_{ud}^* Q_2^{c\bar{u}d} \,, \qquad Q_2^{c\bar{u}d} \;=\; (\overline{d}_L^{\alpha}\gamma_\mu u_L^{\alpha})(\overline{c}_L^{\beta}\gamma^\mu b_L^{\beta}) \,.$$

The colors are contracted as shown since the W boson exchanged between the $(\overline{d}_L^{\alpha}\gamma_\mu u_L^{\alpha})$ and $(\overline{c}_L^{\beta}\gamma^\mu b_L^{\beta})$ does not carry color.

Effect of Strong Interactions

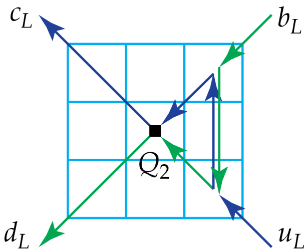
- Scale relevant for B-decay is $\mu = m_b$.
- Since $m_b \ll M_W$, we need an OPE of the interaction Hamiltonian at $\mu = m_b$ in terms of the "local" operators defined at $\mu = M_W$.
- Gluon exchange induces a new operator:



$$Q_1^{c\bar{u}d} = (\overline{d}_L^{\alpha}\gamma_\mu u_L^{\beta})(\overline{c}_L^{\beta}\gamma^\mu b_L^{\alpha}) , \quad Q_2^{c\bar{u}d} = (\overline{d}_L^{\alpha}\gamma_\mu u_L^{\alpha})(\overline{c}_L^{\beta}\gamma^\mu b_L^{\beta}) .$$

Effect of Strong Interactions

- As we lower the renormalization scale μ from $\mu = M_W$ to $\mu = m_b$, the two operators will mix.



- The Wilson coefficients in Eq. (62) of the **Belle II Physics Book** describe these effects:

$$H^{b\to c\bar{u}d}(m_b) \;=\; \frac{4G_F}{\sqrt{2}} V_{cb}V_{ud}^{*} \sum_{j=1,2} C_j(m_b)\, Q_j^{c\bar{u}d}\;.$$

Effect of Strong Interactions

- The Wilson coefficients can be calculated perturbatively since the QCD coupling $g_s(\mu)$ is small in the range $m_b \leq \mu \leq M_W$. (It becomes large at around $\mu = \Lambda_{\mathrm{QCD}} \approx 300\,\mathrm{MeV}$.)
- Eq. (63) of the **Belle II Physics Book** shows how the Wilson coefficients at $\mu = M_W$ and $\mu = m_b$ are related:

$$\vec{C}(\mu_b) \;=\; U(\mu_b, \mu_W)\vec{C}(\mu_W)\;,$$

where $\vec{C}(\mu)$ is given in Eq. (64) :

$$\vec{C}(\mu) \;=\; \begin{pmatrix} C_1(\mu) \\ C_2(\mu) \end{pmatrix}$$

$U(\mu_b, \mu_W)$ is a 2×2 matrix. Note that

$$\vec{C}(\mu_W) \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Expression for $U(\mu_b, \mu_W)$ does not seem to be given in the **Belle II Physics Book**.

New Physics Effects

- Eq. (417) of the **Belle II Physics Book** shows operators that lead to $c \to u\ell^+\ell^-$ $(D^0 \to \ell^+\ell^-)$:

$$\mathcal{H}_{\mathrm{NP}}^{\mathrm{rare}} = \sum_{i=1}^{10} \frac{\tilde{C}_i(\mu)}{\Lambda^2}\, \tilde{\mathcal{Q}}_i \,,$$

where

$$\begin{aligned}
\tilde{\mathcal{Q}}_1 &= (\bar{\ell}_L \gamma_\mu \ell_L)(\overline{u}_L \gamma^\mu c_L)\,, & \tilde{\mathcal{Q}}_2 &= (\bar{\ell}_L \gamma_\mu \ell_L)(\overline{u}_R \gamma^\mu c_R)\,,\\
\tilde{\mathcal{Q}}_3 &= (\bar{\ell}_L \ell_R)(\overline{u}_R c_L)\,, & \tilde{\mathcal{Q}}_4 &= (\bar{\ell}_R \ell_L)(\overline{u}_R c_L)\,,\\
\tilde{\mathcal{Q}}_5 &= (\bar{\ell}_R \sigma_{\mu\nu} \ell_L)(\overline{u}_R \sigma^{\mu\nu} c_L)\,.
\end{aligned}$$

$\tilde{\mathcal{Q}}_{6\sim10}$ are obtained from $\tilde{\mathcal{Q}}_{1\sim5}$ via the interchange $L \leftrightarrow R$.
These operators do not exist or are highly suppressed within the SM.

New Physics Effects

$$\mathcal{H}_{\rm NP}^{\rm rare} = \sum_{i=1}^{10} \frac{\tilde{C}_i(\mu)}{\Lambda^2}\,\tilde{\mathcal{Q}}_i \;,$$

- Λ is the scale of new particles that mediate the interactions.
- Since we are considering charm decay, we should choose $\mu = m_c$, but that is not important here since we cannot calculate them anyway in the absence of a model that predicts these operators. So $C_i(\mu)$ are just arbitrary constants.

Summary

- OPE's appear whenever we need to compare the behavior of a QFT at different renormalization scales μ.
- The Wilson coefficients track how the operators evolve with μ.
- OPE is used to match terms in QFT's that are effective at different scales.
- We can now understand what all those yucky equations in the **Belle II Physics Book** mean. Hurray! :)