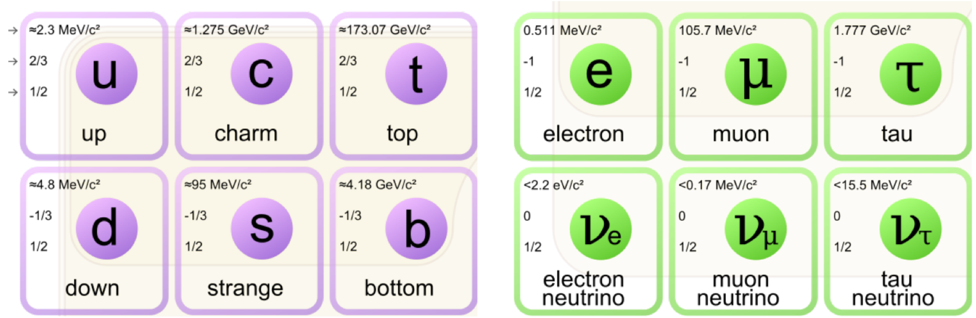
Time-Dependent Analysis Formulations

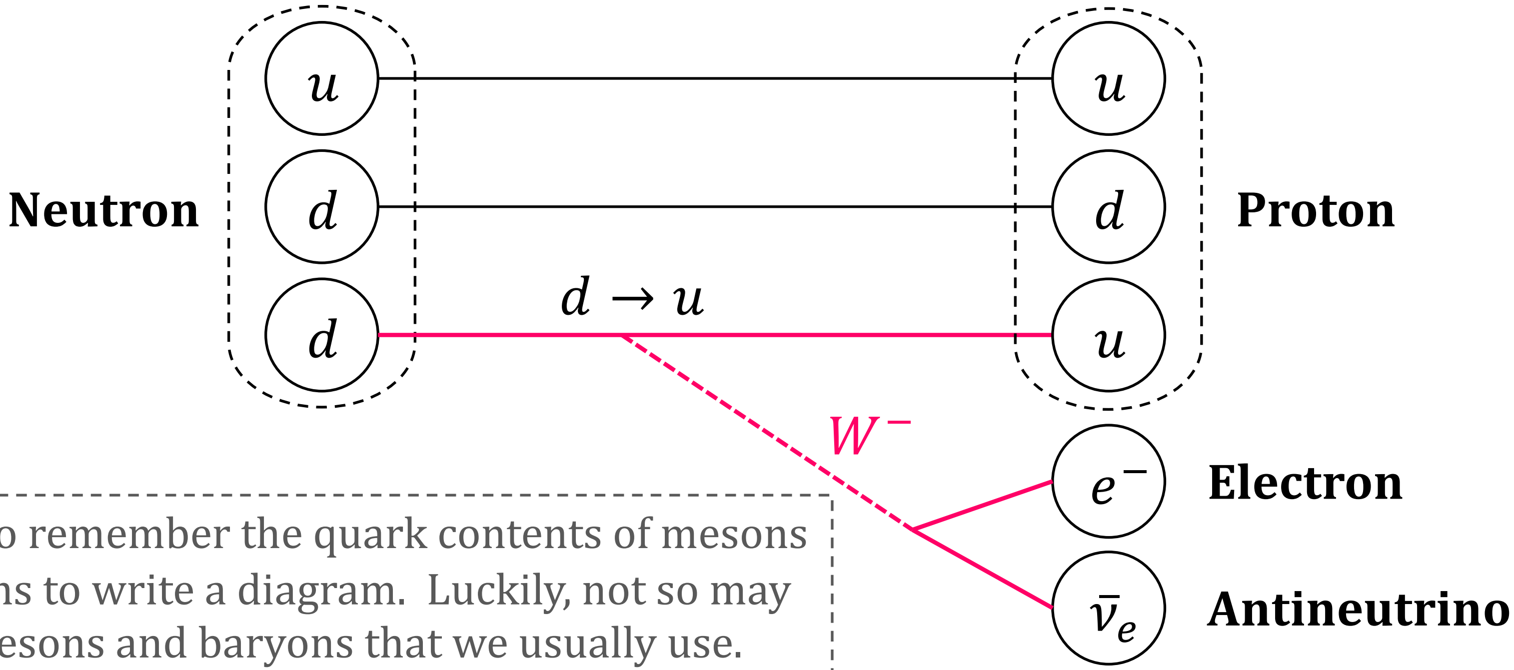
Takeo Higuchi

Kavli IPMU, the University of Tokyo

Weak Interaction



Neutron β decay



You need to remember the quark contents of mesons and baryons to write a diagram. Luckily, not so may kinds of mesons and baryons that we usually use.

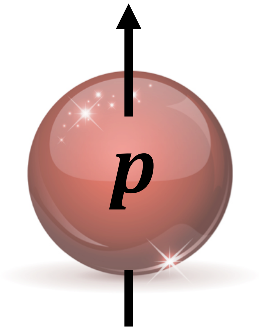
π^+ decay – process to generate the secondary cosmic ray μ^+

History of the Weak Interaction

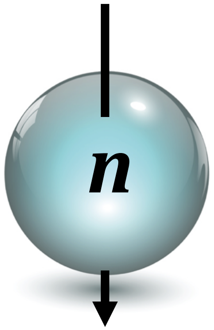
- **1919, 1932**: discoveries of proton and neutron by E. Rutherford and J. Chadwick, respectively.

E. Rutherford

J. Chadwick

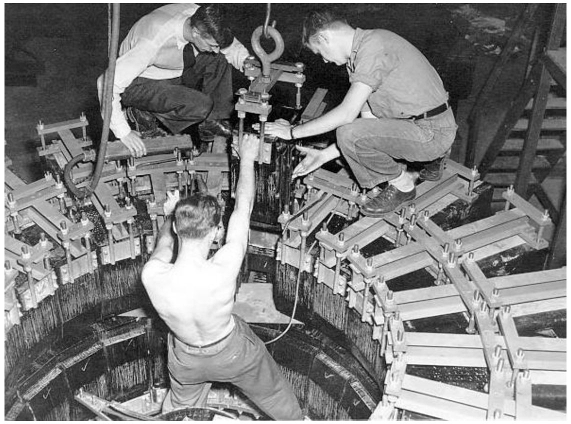


Isospin up



Isospin down

- **~1940s**: recovery of the accelerator operation from the suspension by WWII and discoveries of tons of *new* particle with it.



The list of "elementary" particles:

Before 1940s: p, n, e^- + some exceptions

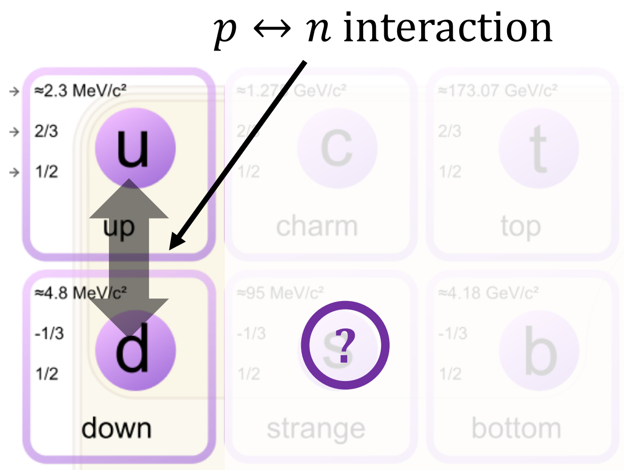
After 1940s: $p, n, e^-, \pi^0, \eta, \eta', K^+, K^-,$
$\Delta^{++}, \Delta^+, \Delta^0, \Xi^+, \Xi^0, \Sigma^+, \ldots$

TOO MUCH: "it is hard to believe they are really the elementary particles."

History of the Weak Interaction

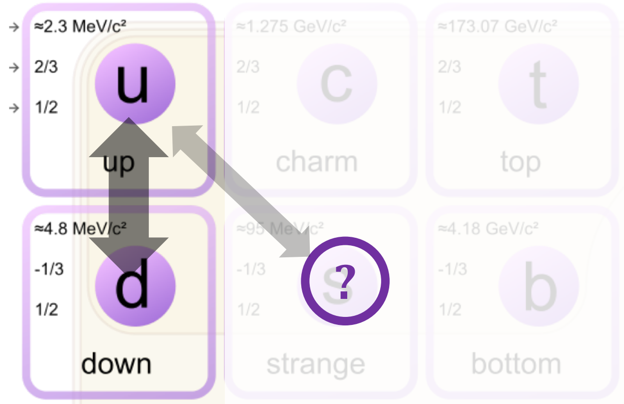
- **1953**: M. Gell-Mann hypothesized that all of those “new” particles are a composition of more fundamental 3 kinds of sub-particles: up, down, and strange quarks.

M. Gell-Mann



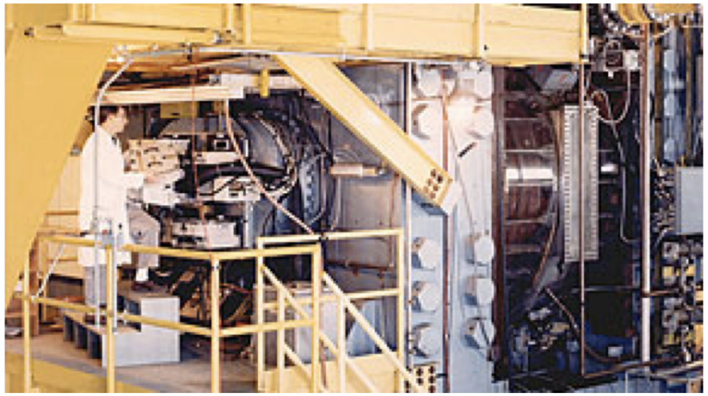
- **1963**: N. Cabibbo proposed a new particle interaction as a solution to the puzzle of strange particle decay to other particles

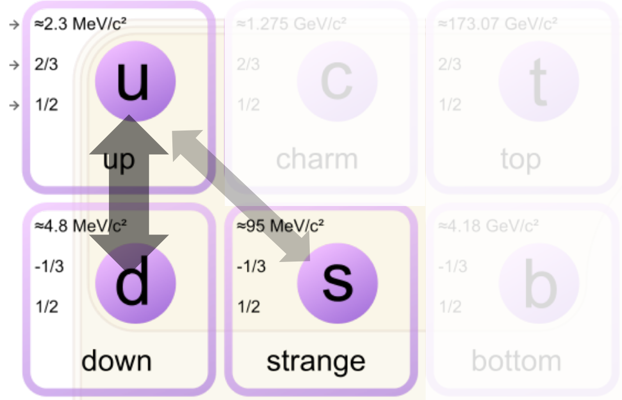
N. Cabibbo



History of the Weak Interaction

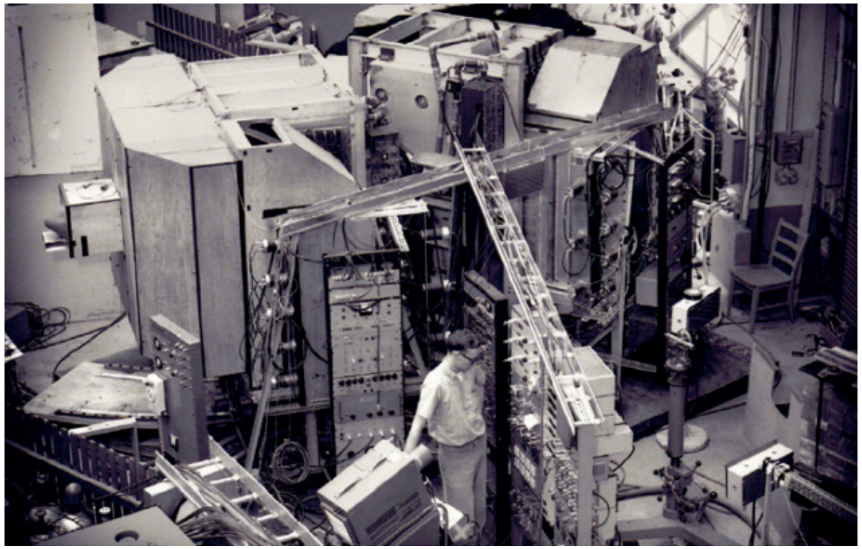
- **1964:** A new baryon $\Omega^-(sss)$ was discovered in BNL, whose existence was theoretically predicted by the quark model.





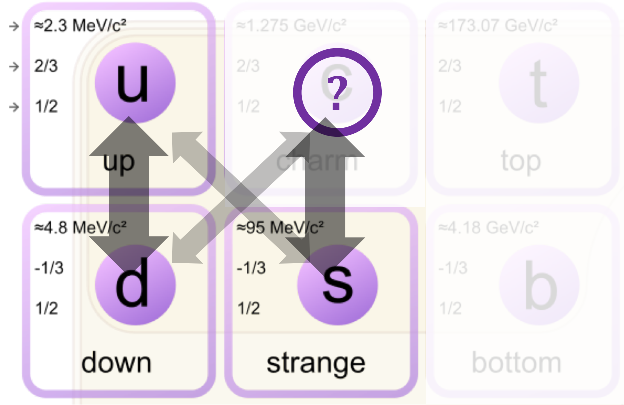
- **1964:** J. W. Cronin *et al*. discovered the *CP* violation by in the neutral *K*-meson system with an experimental setup in BNL.

J. W. Cronin



History of the Weak Interaction

- **1970:** S. L. Glashow, J. Iliopoulos, and L. Maiani proposed a new theory as a solution to the puzzle of the $K_L^0 \to \mu^+\mu^-$ decay suppression. A new quark "charm" and a new interaction $d \leftrightarrow c$ were introduced.



- When we simply say "quarks" and "leptons", what eigenstates are implied?

  ➡ **Mass** eigenstates

- When we say "quark" and "lepton" exchanges, what eigenstates are implied?

  ➡ **Weak-interaction** eigenstates

S. Glashow

J. Iliopoulos

L. Maiani

$$(\bar{u}^{(w)}\ \ \bar{c}^{(w)})\gamma^\mu \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}d^{(w)}\\ s^{(w)}\end{pmatrix}$$

$$(\bar{u}^{(m)}\ \ \bar{c}^{(m)})\gamma^\mu \left(U^u_{(m)\to(w)}\right)^\dagger \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\left(U^d_{(m)\to(w)}\right)\begin{pmatrix}d^{(m)}\\ s^{(m)}\end{pmatrix}$$

Change of the basis from (mass) to (weak interaction)

$$(\bar{u}^{(m)}\ \ \bar{c}^{(m)})\gamma^\mu \begin{pmatrix}V_{ud} & V_{us}\\ V_{cd} & V_{cs}\end{pmatrix}\begin{pmatrix}d^{(m)}\\ s^{(m)}\end{pmatrix}$$

GIM mechanism

The annoying superscript (m) can be dropped.

Discrete Symmetry, *C*, *P*, *T*

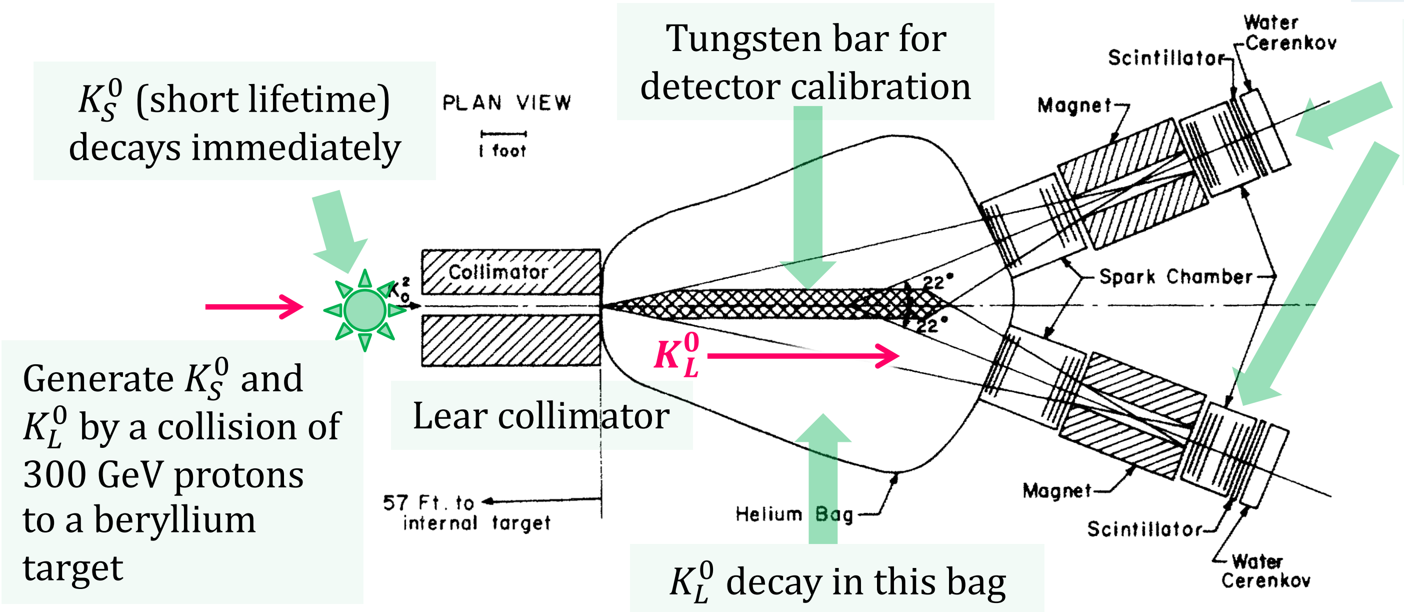
Transformation	Op	Note
Parity	P	**Transform the wavefunction position from x to $-x$** Discovered by C. S. Wu *et al*. in the beta decay of ^{60}Co in 1956. Corresponds to the fact that only LH'ed particles participate in the weak interaction.
Charge conjugation	C	**Transform the internal quantum numbers of the particle** Flips the sign of the electric charge, lepton number, and baryon number of the particle, but conserves the E, $\vec{p}$, m, and **spin**.
Time reversal	T	**Transform the wavefunction time from t to $-t$** Discovered by the CPLEAR experiment in the neutral K-meson system in 1998.
Matter anti-matter	CP	**Transform a particle to its anti-particle partner** Discovered by J. W. Cronin *et al*. in the K_L^0 decay in 1964.
All of them	CPT	**Product of the *P*, *C*, and *T*** Derived from Wightman's axioms. Ensures $m_A = m_{\bar{A}}$ and $\tau_A = \tau_{\bar{A}}$ together with the Bose/Fermi quantum statistics and other fundamental theorems. No evidence for the CPT violation.

Discovery of the *CP* Violation

- It was assumed that the mass eigenstate K_L^0 (observable particle) is also a CP eigenstate with the eigenvalue $\eta_{CP} = -1$ before 1964. J. W. Cronin *et al*. doubted this assumption and conducted an experiment to test it in 1964.

Neutral K meson properties

	Mass (MeV/c^2)	Lifetime (ps)	CP	Major decay modes
K_S^0	498	0.090	+1 (?)	$\pi^+\pi^-, \pi^0\pi^0$
K_L^0		51	−1 (?)	$\pi^+\pi^-\pi^0, \pi^0\pi^0\pi^0$



J. W. Cronin

- They observed 45 ± 9 $K_L^0 \to \pi^+\pi^-$ decays ($\eta_{CP} = +1$ decay). It turned out that **the K_L^0 (mass eigenstate) is not a *CP* eigenstate ... *CP* violation.**

J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Lett. **13**, 138 (1964).

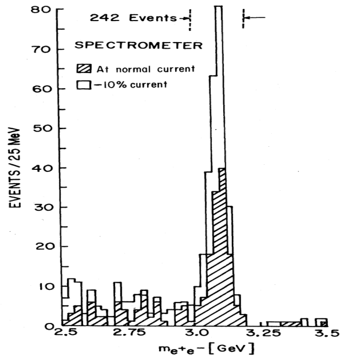
History of the Weak Interaction

M. Kobayashi, T. Maskawa

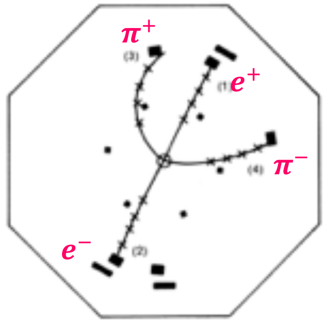
- **1973**: M. Kobayashi, T. Maskawa proposed a new theory to explain the *CP* violation discovered by J. W. Cronin *et al*. They predicted the number of quark kinds is >6 kinds only 3 kinds of them were discovered.

- **1974**: S. Ting and B. Richter made independent discoveries of the charm quark.

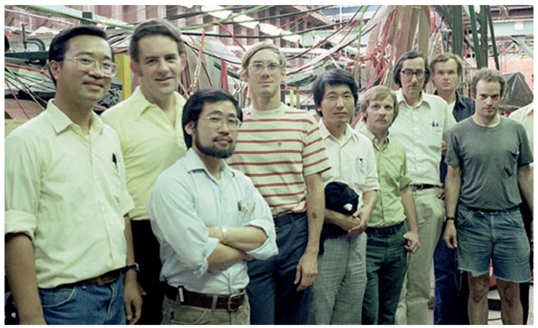
S. C. Ting

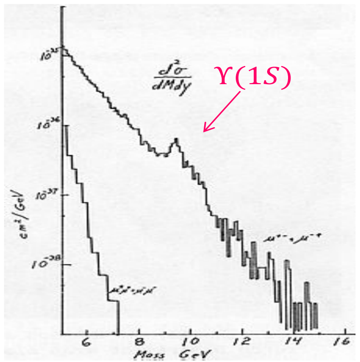


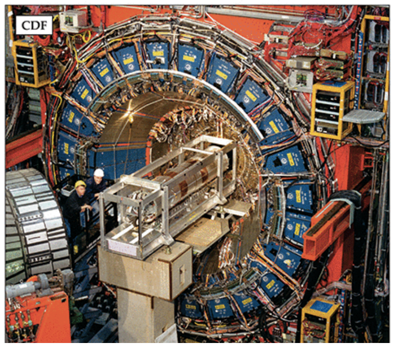
B. Richter

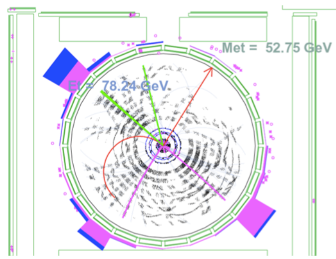


- **1977, 1995**: the bottom and top quarks were discovered by expts. in Fermilab.









Kobayashi-Maskawa Theory

- GIM mechanism $j^{\mu} = -i\frac{g_W}{\sqrt{2}} \cdot (\bar{u} \;\; \bar{c})\, \gamma^{\mu} \begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$

Quark flavor change from y to x

$$j^{\mu}_{y \to x} \propto \boldsymbol{V_{xy}} \frac{g_W}{\sqrt{2}}$$

Anti-quark flavor change from $\bar{y}$ to $\bar{x}$

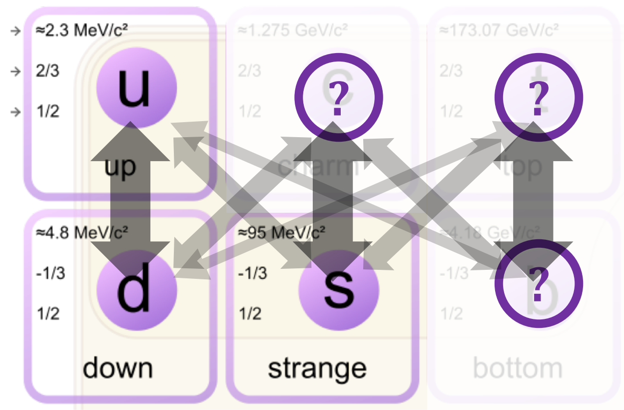
$$j^{\mu}_{\bar{y} \to \bar{x}} \propto \boldsymbol{V^*_{xy}} \frac{g_W}{\sqrt{2}}$$

- **When $V_{xy} \neq V^*_{xy}$ (i.e.: $\arg(V_{xy}) \neq 0$), the CP symmetry does not conserve.**
- By extending the number of quark generations from 2 to 3 the matrix element may become complex.

Kobayashi-Maskawa (3 generations)

$$(\bar{u} \;\; \bar{c} \;\; \bar{t})\, \gamma^{\mu} \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

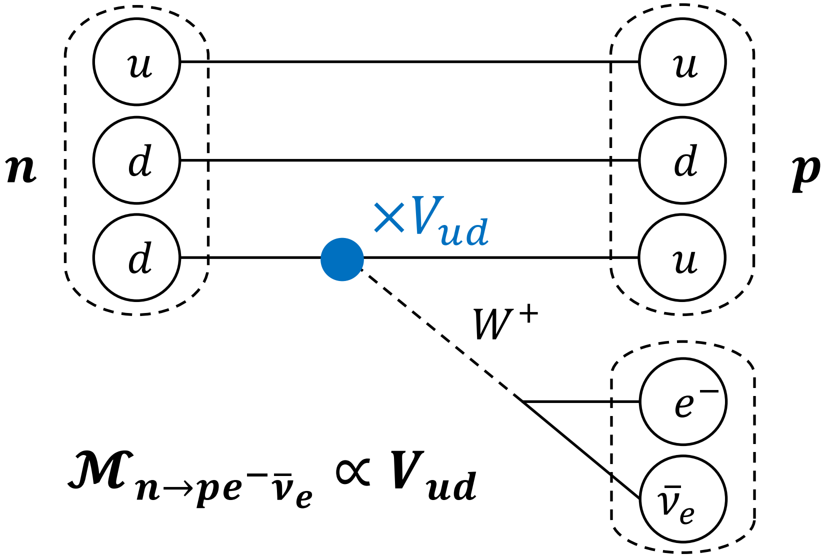
CKM matrix



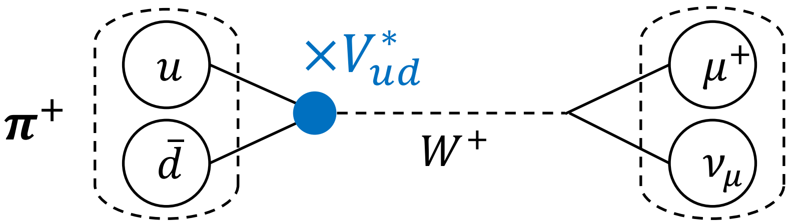
In general, an $n \times n$ unitary matrix has $n^2 - n(n-1)/2 - (2n-1)$ irreducible complex numbers that cannot be removed by phase redefinition. When $n \geq 3$, $n^2 - n(n-1)/2 - (2n-1) \geq 1$.

Feynman Diagram Examples

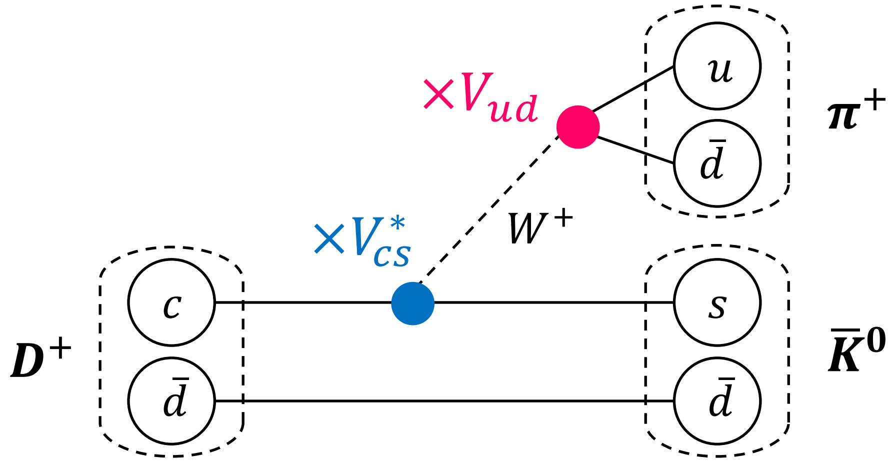
The strength (amplitude $\mathcal{M}$) is proportional to …



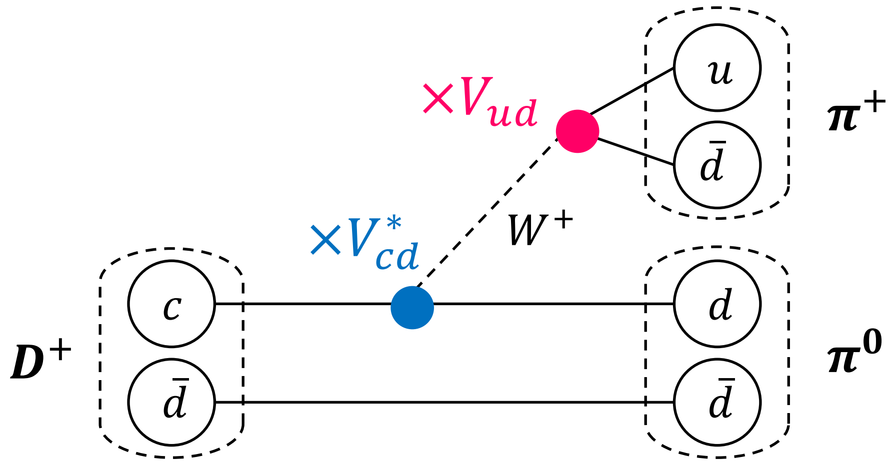
$$\mathcal{M}_{n\to pe^-\bar{\nu}_e} \propto V_{ud}$$



$$\mathcal{M}_{\pi^+\to\mu^+\nu_\mu} \propto V_{ud}^*$$



$$\mathcal{M}_{D^+\to\bar{K}^0\pi^+} \propto V_{cs}^* V_{ud}$$



$$\mathcal{M}_{D^+\to\pi^0\pi^+} \propto V_{ud}^* V_{ud}$$

CKM Matrix

- The CKM matrix is a unitary matrix: $\begin{pmatrix} V_{ud} & V_{us} & \boldsymbol{V_{ub}} \\ V_{cd} & V_{cs} & V_{cb} \\ \boldsymbol{V_{td}} & V_{ts} & V_{tb} \end{pmatrix}^\dagger \begin{pmatrix} V_{ud} & V_{us} & \boldsymbol{V_{ub}} \\ V_{cd} & V_{cs} & V_{cb} \\ \boldsymbol{V_{td}} & V_{ts} & V_{tb} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

  From the unitarity condition, 6 equations are derived.

  (a) $V_{ud}V_{us}^* + V_{cd}V_{cs}^* + V_{td}V_{ts}^* = 0$

  (b) $V_{ud}V_{cd}^* + V_{us}V_{cs}^* + V_{ub}V_{cb}^* = 0$

  (c) $V_{us}V_{ub}^* + V_{cs}V_{cb}^* + V_{ts}V_{tb}^* = 0$

  (d) $V_{cd}V_{td}^* + V_{cs}V_{ts}^* + V_{cb}V_{tb}^* = 0$

  (e) $V_{ud}V_{td}^* + V_{us}V_{ts}^* + V_{ub}V_{tb}^* = 0$

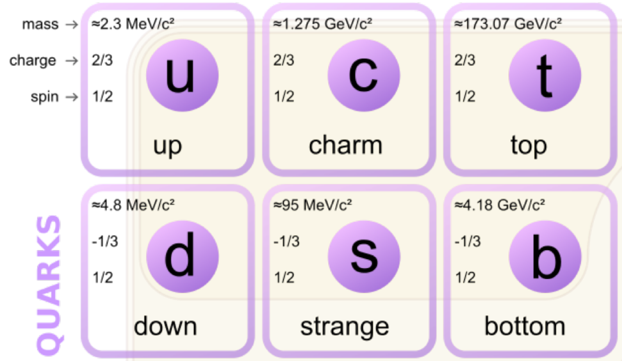
  (f) $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$

- From physics discussion, the Wolfenstein parameterization is obtained:

$$V_{\mathrm{CKM}} \equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - \boldsymbol{i\eta}) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-\boldsymbol{i\eta}) & -A\lambda^2 & 1 \end{pmatrix}$$

  - You need to remember that V_{td} and V_{ub} are complex.
  - You need to remember $\lambda \approx 0.2$ plus the order of λ for each element.
  - You need to remember $A \approx 0.8$.

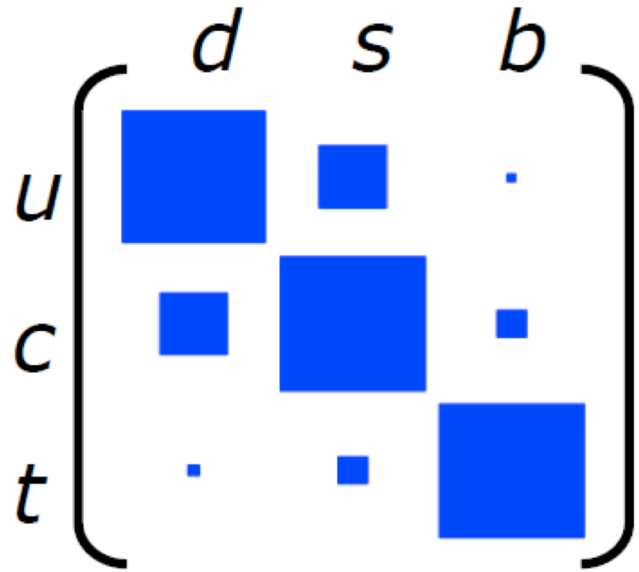
CKM Matrix

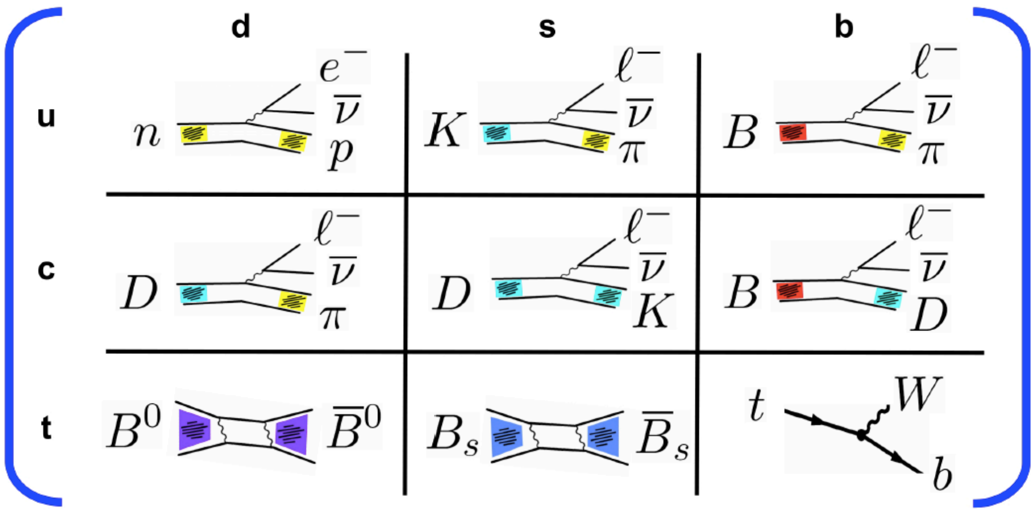


PDG2020

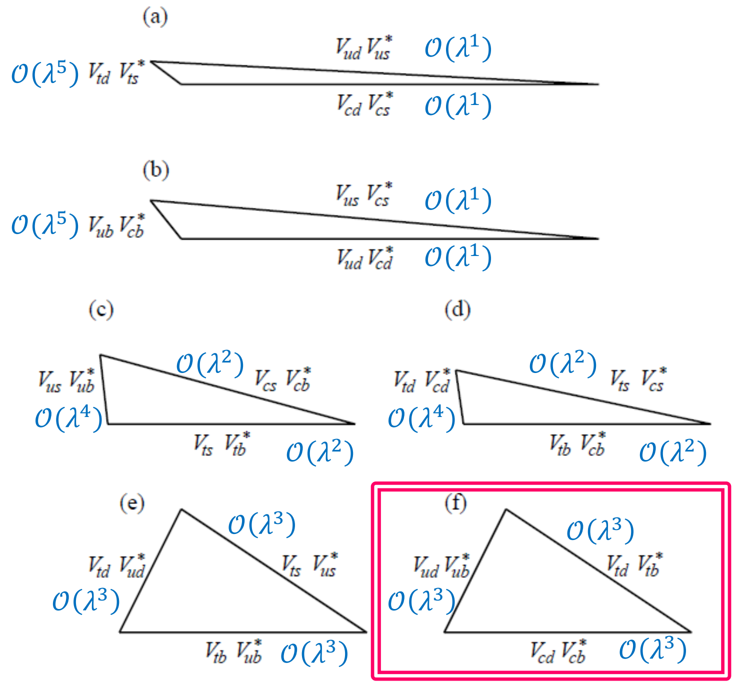
$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$= \begin{pmatrix} 0.97370 \pm 0.00014 & 0.2245 \pm 0.0008 & (3.82 \pm 0.24)\times 10^{-3} \\ 0.221 \pm 0.004 & 0.987 \pm 0.011 & (41.0 \pm 1.4)\times 10^{-3} \\ (8.0 \pm 0.3)\times 10^{-3} & (38.8 \pm 1.1)\times 10^{-3} & 1.013 \pm 0.030 \end{pmatrix}$$

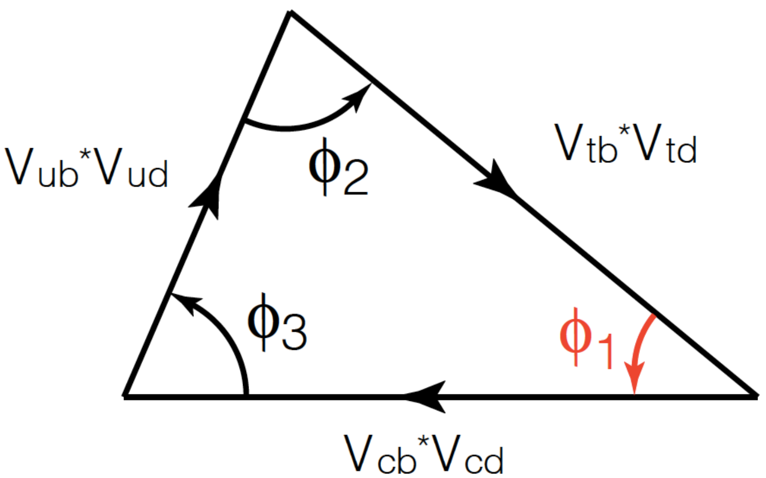




CKM Triangle



- Each of the equation forms a triangle on the complex plane.
- The bottom right triangle, which is associated to the equation $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$ is moderately large.
- By assuming $V_{ud}V_{ub}^*$, $V_{cd}V_{cb}^*$, and $V_{td}V_{tb}^*$ are vectors, we can draw a triangle associated to the equation on the complex plane, which is called "CKM triangle".



Interior angle definition

$$\phi_1 \equiv \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right) = \pi - \arg(V_{td})$$

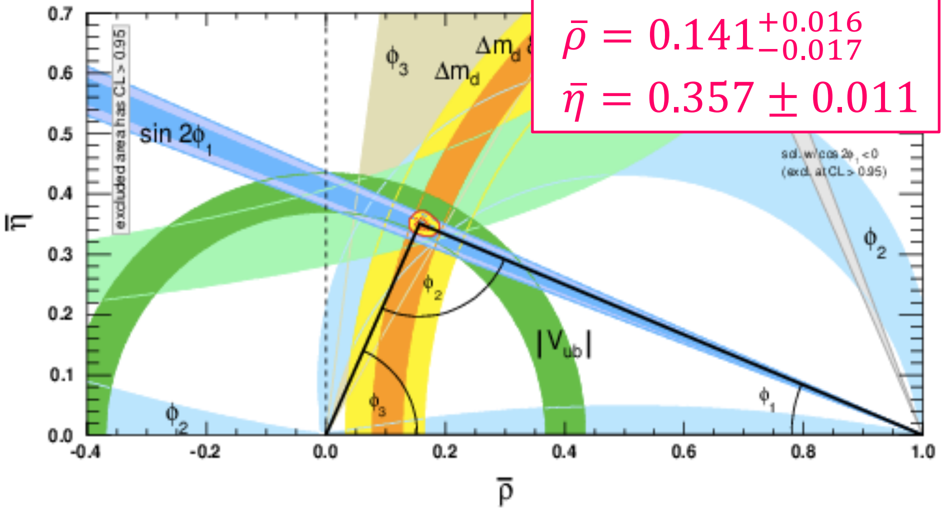
$$\phi_2 \equiv \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right)$$

$$\phi_3 \equiv \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right)$$

If the KM theory is correct, $\phi_1 \neq 0, \pi$.

CKM Triangle – Current Status

CKMfitter 2019, PDG2020



Apex position

$$\bar{\rho} = 0.141^{+0.016}_{-0.017}$$
$$\bar{\eta} = 0.357 \pm 0.011$$

$$\phi_1 \equiv \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right) = (22.56^{+0.47}_{-0.40})^\circ$$

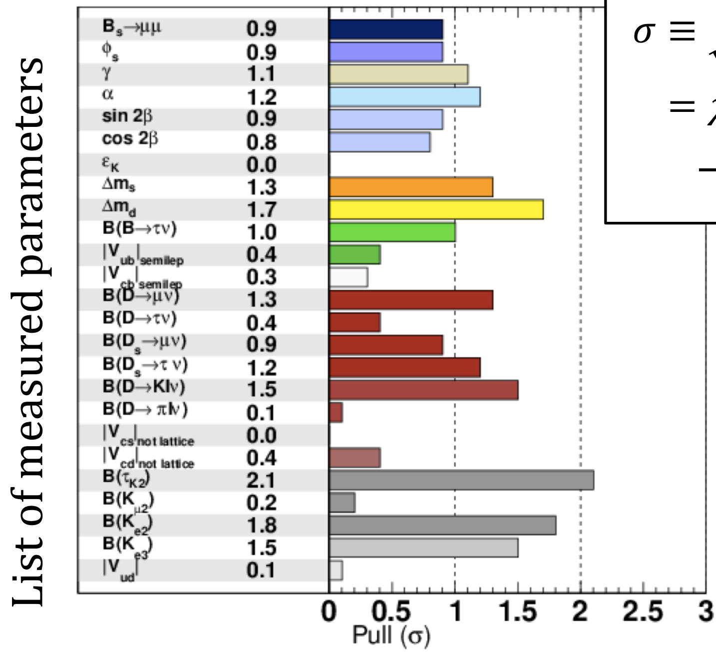
$$\phi_2 \equiv \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right) = (91.7^{+1.7}_{-1.1})^\circ$$

$$\phi_3 \equiv \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) = (65.8^{+0.94}_{-1.29})^\circ$$

$$|V_{cb}| = 0.04162^{+0.00026}_{-0.00080}$$

$$|V_{ub}| = 0.003683^{+0.000075}_{-0.000061}$$

PDG 2020



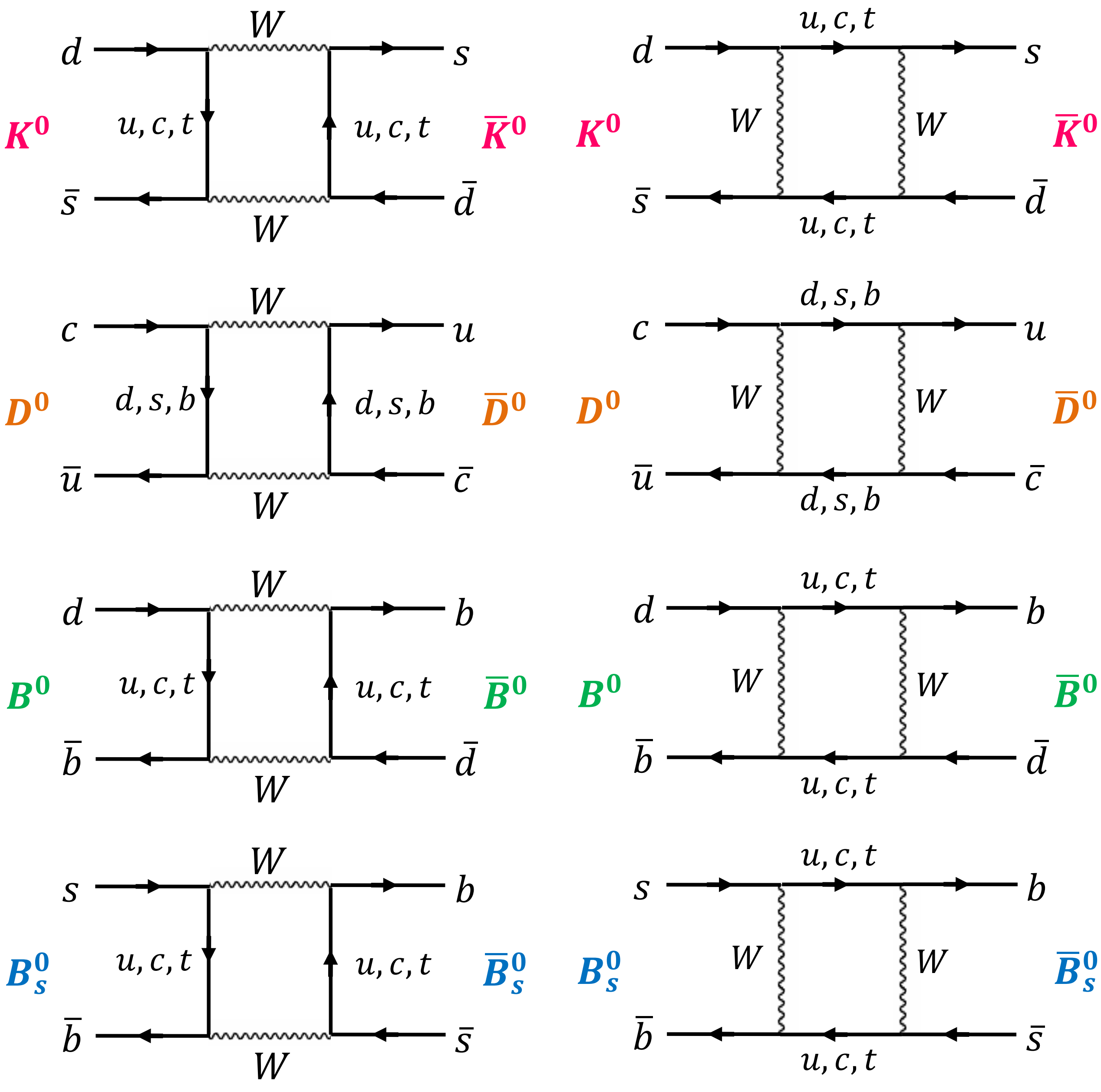
$$\sigma \equiv \sqrt{\chi^2_{x;\mathrm{min}}}$$
$$= \chi^2_{\text{with data on } X} - \chi^2_{\text{without data on } X}$$

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9985 \pm 0.0005$$
$$|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1.025 \pm 0.020$$
$$|V_{ud}|^2 + |V_{cd}|^2 + |V_{tb}|^2 = 0.9970 \pm 0.0018$$
$$|V_{us}|^2 + |V_{cs}|^2 + |V_{ts}|^2 = 1.026 \pm 0.022$$

The unitarity of the CKM matrix holds surprisingly well (except the first relation).

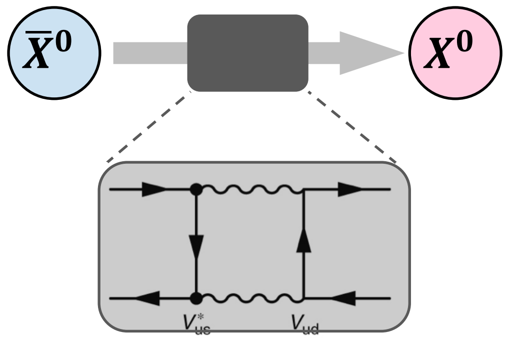
X^0-$\bar{X}^0$ Mixing, $X^0 = K^0, D^0, B^0, B_s^0, \ldots$

K^0 ... Cronin *et al.*
D^0, B^0 ... Belle, Belle II, BaBar, LHCb
B_s^0 ... LHCb



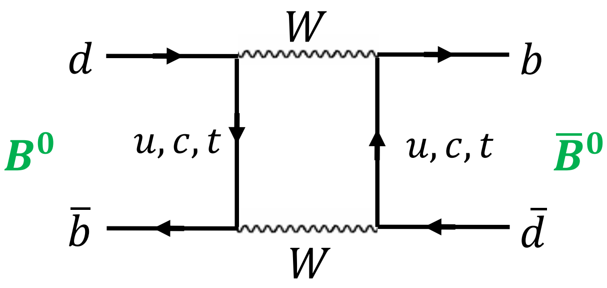
Even if the neutral meson is initially X^0, the probability to find the particle in the $\bar{X}^0$ state (*CP* partner of the X^0) is non zero after a certain time *t* because of the process called

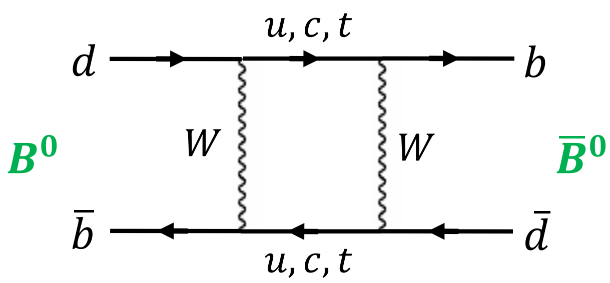
X^0-$\bar{X}^0$ Mixing



Note: nothing relevant to the pair creation!

B^0-$\bar{B}^0$ Mixing





Each box diagram has $V_{td} \times V_{td}$.

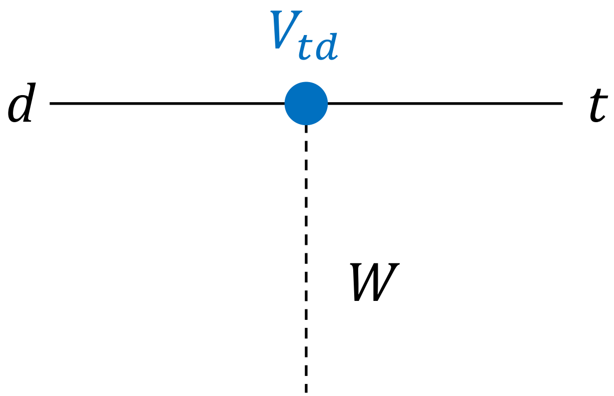
By using the B^0-$\bar{B}^0$ mixing mechanism, we can measure the V_{td}^2 phase.

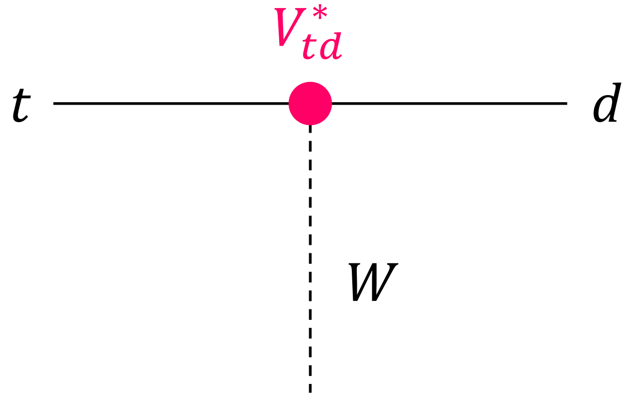
- **Does the B^0 meson really exist?**
- **Does the B^0-$\bar{B}^0$ mixing really exist?**

Test of the KM Theory: Measurement of V_{td}

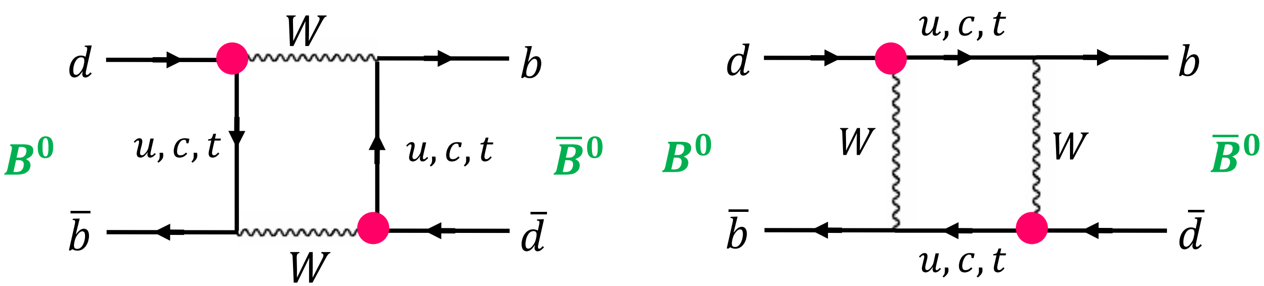
- **Hypothesize that V_{td} (and V_{ub}) are complex.**

- **How do we measure the V_{td} phase, $\arg(V_{td})$?**
  → should use particle interactions that contain V_{td} like these:





- **The B^0-$\bar{B}^0$ mixing is a useful phenomenon to access $\arg(V_{td})$.**

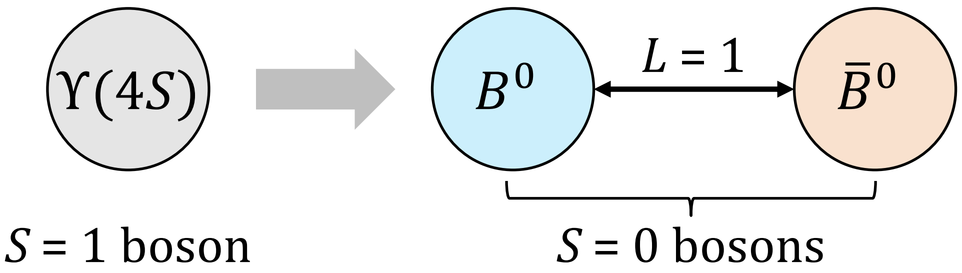


$\propto V_{td}^2$

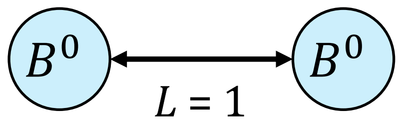
B^0-$\bar{B}^0$ Coherence

- Just *forget* about the B^0-$\bar{B}^0$ mixing.
- When B^0 and $\bar{B}^0$ are pair-produced from the $\Upsilon(4S) \to B^0\bar{B}^0$ decay, the two B-mesons take neither (B^0, B^0) nor $(\bar{B}^0, \bar{B}^0)$ state but only the $(B^0, \bar{B}^0)$ state.

[proof]



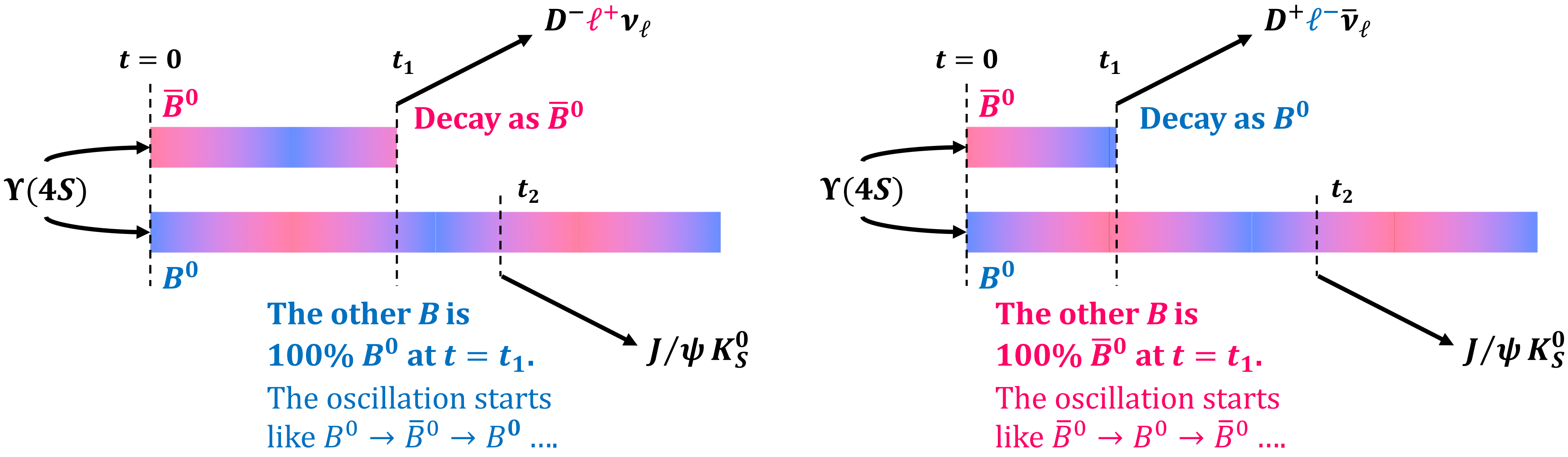
$\Upsilon(4S)$ is a S = 1 boson, and B^0, $\bar{B}^0$ are S = 0 boson. Because of the angular momentum conservation, the orbital angular momentum L between the two B mesons is L = 1.



If both of the two B mesons take the same particle state B^0, the wavefunction of the system is exchange-symmetric of the two B mesons because of the Bose-Einstein statistics. However, the wavefunction must be exchange-antisymmetric of two same particles with L = 1. These two statements are inconsistent. The same inconsistency is true for the $\bar{B}^0$- $\bar{B}^0$ system.

Thence, **only the $(B^0, \bar{B}^0)$ state is allowed**. □

B^0-$\bar{B}^0$ Mixing And B^0-$\bar{B}^0$ Coherence



- The flavor of the two B mesons can be known only stochastically until they decay to some other particles. **... B^0-$\bar{B}^0$ mixing**

- One of the two B mesons B_1 decays to a flavor specific state. The B_1 flavor can be known from the decay products.

- The flavor of the other B meson B_2 at the time of the B_1 decay t=t_1 is opposite to the B_1 flavor. **... B^0-$\bar{B}^0$ coherence**

- The B_2 flavor can be known only statistically after t_1 until it decays to some other particles. **... B^0-$\bar{B}^0$ mixing**

CP Violation in *B* Decays

I. I. Bigi

A. I. Sanda

Phenomenologist who theoretically developed the measurement procedure of the *CP* violation in *B* decays.

B^0-$\bar{B}^0$ Mixing

- A meson initially in the B^0 state $\left|\boldsymbol{B^0}\right\rangle \xrightarrow{\text{after a time } t} \left|\boldsymbol{B^0}\,(\boldsymbol{t})\right\rangle = ☺ \left|\boldsymbol{B^0}\right\rangle + ☹ \left|\boldsymbol{B^0}\right\rangle$

  ☺ $\neq 1$ and ☹ $\neq 0$ is the essence of the B^0-$\bar{B}^0$ mixing

- A meson initially in the $\bar{B}^0$ state $\left|\boldsymbol{\bar{B}^0}\right\rangle \xrightarrow{\text{after a time } t} \left|\boldsymbol{\bar{B}^0}\,(\boldsymbol{t})\right\rangle = ♤ \left|\boldsymbol{B^0}\right\rangle + ♡ \left|\boldsymbol{\bar{B}^0}\right\rangle$

  ♤ $\neq 0$ and ♡ $\neq 1$ is the essence of the B^0-$\bar{B}^0$ mixing

How do we express ☺ *etc*. as a function of *t* in general?

- Remember quantum mechanics:
  discussion on the time propagation of a state
  → discussion on the energy of the state → discussion on the mass of the state.
- As J. W. Christenson demonstrated, the $\left|X^0\right\rangle$ and $\left|\bar{X}^0\right\rangle$ are a *CP* eigenstate but not a mass eigenstate.
- **Relate the *CP* eigenstate to a mass eigenstates by force.**

B^0-$\bar{B}^0$ Mixing

- $|B(t)\rangle = ☺(t)|B\rangle + ☹(t)|\bar{B}^0\rangle \equiv \alpha(t)\begin{pmatrix}1\\0\end{pmatrix} + \beta(t)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix}$

  $|B^0\rangle$ $|\bar{B}^0\rangle$

- The phenomenological time-dependent Schrödinger equations are:

$$i\frac{\partial}{\partial t}\begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix} = \mathcal{H}\begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix} \Rightarrow i\frac{\partial}{\partial t}\begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix} = \left(M - \frac{i}{2}\Gamma\right)\begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix}$$

$X^0, \bar{X}^0$ mass and decay

$$\equiv \begin{pmatrix}M_{11} - i\Gamma_{11}/2 & M_{12} - i\Gamma_{12}/2\\ M_{21} - i\Gamma_{21}/2 & M_{22} - i\Gamma_{22}/2\end{pmatrix}\begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix}$$

Because the operators M and Γ are Hermitian, $M_{21} = M_{12}^*$ and $\Gamma_{21} = \Gamma_{12}^*$.
Then, because of the CPT theorem, $M_{11} = M_{22} \equiv M$ and $\Gamma_{11} = \Gamma_{22} \equiv \Gamma$.

$$i\frac{\partial}{\partial t}\begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix} = \begin{pmatrix}M - i\Gamma/2 & M_{12} - i\Gamma_{12}/2\\ M_{12}^* - i\Gamma_{12}^*/2 & M - i\Gamma/2\end{pmatrix}\begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix}$$

- If $M_{12} - i\Gamma_{12}/2 = M_{12}^* - i\Gamma_{12}^*/2 = 0 \rightarrow$ no B^0-$\bar{B}^0$ mixing; otherwise, the B^0-$\bar{B}^0$ mixing occurs and $|B(t)\rangle$ and $|\bar{B}(t)\rangle$ are no longer the mass eigenstates.

B^0-$\bar{B}^0$ Mixing

- Let the eigenvectors of $\left(M - \frac{i}{2}\Gamma\right) = \begin{pmatrix} M - i\Gamma/2 & M_{12} - i\Gamma_{12}/2 \\ M_{12}^* - i\Gamma_{12}^*/2 & M - i\Gamma/2 \end{pmatrix}$ and the associated eigenvalues to be $|B_H\rangle, \mu_H$ and $|B_L\rangle, \mu_L$. These definitions lead

  $|B_H(t)\rangle = \exp(-i\mu_H t)\,|B_H\rangle$ and $|B_L(t)\rangle = \exp(-i\mu_L t)\,|B_L\rangle$.

- On the other hand, by an arithmetic calculation, we obtain

  $|B_H\rangle = p|B^0\rangle - q|\bar{B}^0\rangle,\ \mu_H = M - i\Gamma/2 - \sqrt{(M_{12} - i\Gamma_{12}/2)(M_{12}^* - i\Gamma_{12}^*/2)}$ and

  $|B_L\rangle = p|B^0\rangle + q|\bar{B}^0\rangle,\ \mu_L = M - i\Gamma/2 + \sqrt{(M_{12} - i\Gamma_{12}/2)(M_{12}^* - i\Gamma_{12}^*/2)}$,

  where $\frac{p}{q} = \sqrt{\frac{M_{12}^* - i\Gamma_{12}^*/2}{M_{12} - i\Gamma_{12}/2}}$ and $|p|^2 + |q|^2 = 1$.

  Additionally, we obtain

  $|B^0\rangle = (|B_L\rangle + |B_H\rangle)/2p$ and $|\bar{B}^0\rangle = (|B_L\rangle - |B_H\rangle)/2q$.

B^0-Meson Decay

- Finally, we obtain

$$|B^0(t)\rangle = \frac{1}{2}\left[(\exp(-i\mu_L t) + \exp(-i\mu_H t))|B^0\rangle + \frac{q}{p}(\exp(-i\mu_L t) - \exp(-i\mu_H t))|\bar{B}^0\rangle\right]$$

$$|\bar{B}^0(t)\rangle = \frac{1}{2}\left[\frac{p}{q}(\exp(-i\mu_L t) - \exp(-i\mu_H t))|B^0\rangle + (\exp(-i\mu_L t) + \exp(-i\mu_H t))|\bar{B}^0\rangle\right]$$

- By defining $f_\pm(t) \equiv \frac{1}{2}(\exp(-i\mu_L t) \pm \exp(-i\mu_H t))$ for brevity, we obtain

$$|B^0(t)\rangle \equiv f_+(t)|B^0\rangle + \frac{q}{p}f_-(t)|\bar{B}^0\rangle \text{ and } |\bar{B}^0(t)\rangle \equiv \frac{p}{q}f_-(t)|B^0\rangle + f_+(t)|\bar{B}^0\rangle .$$

Application: B^0-$\bar{B}^0$ Mixing

- By defining $f_\pm(t) \equiv \frac{1}{2}(\exp(-i\mu_L t) \pm \exp(-i\mu_H t))$ for brevity, we obtain

$$|B^0(t)\rangle \equiv f_+(t)|B^0\rangle + \frac{q}{p} f_-(t)|\bar{B}^0\rangle \text{ and } |\bar{B}^0(t)\rangle \equiv \frac{p}{q} f_-(t)|B^0\rangle + f_+(t)|\bar{B}^0\rangle\ .$$

- Probability to find the particle in the $|B^0\rangle$ and $|\bar{B}^0\rangle$ states at a time t that was initially in the $|B^0\rangle$ state are

$$|\langle B^0|B^0(t)\rangle|^2 = |f_+(t)|^2 = \frac{1}{2}\exp(-\Gamma t)\left[\cosh\left(\frac{\Delta\Gamma_d}{2}t\right) + \cos(\Delta m_d t)\right]$$

$$|\langle \bar{B}^0|B^0(t)\rangle|^2 = \left|\frac{q}{p} f_-(t)\right|^2 = \frac{1}{2}\left|\frac{q}{p}\right|^2 \exp(-\Gamma t)\left[\cosh\left(\frac{\Delta\Gamma_d}{2}t\right) - \cos(\Delta m_d t)\right], \text{ respectively,}$$

where $m_{L,H} \equiv \mathcal{Re}(\mu_{L,H})$, $\Gamma_{L,H} \equiv -2\mathcal{Im}(\mu_{L,H})$, $\Delta m_d \equiv m_H - m_L$, and $\Delta\Gamma_d \equiv \Gamma_H - \Gamma$.

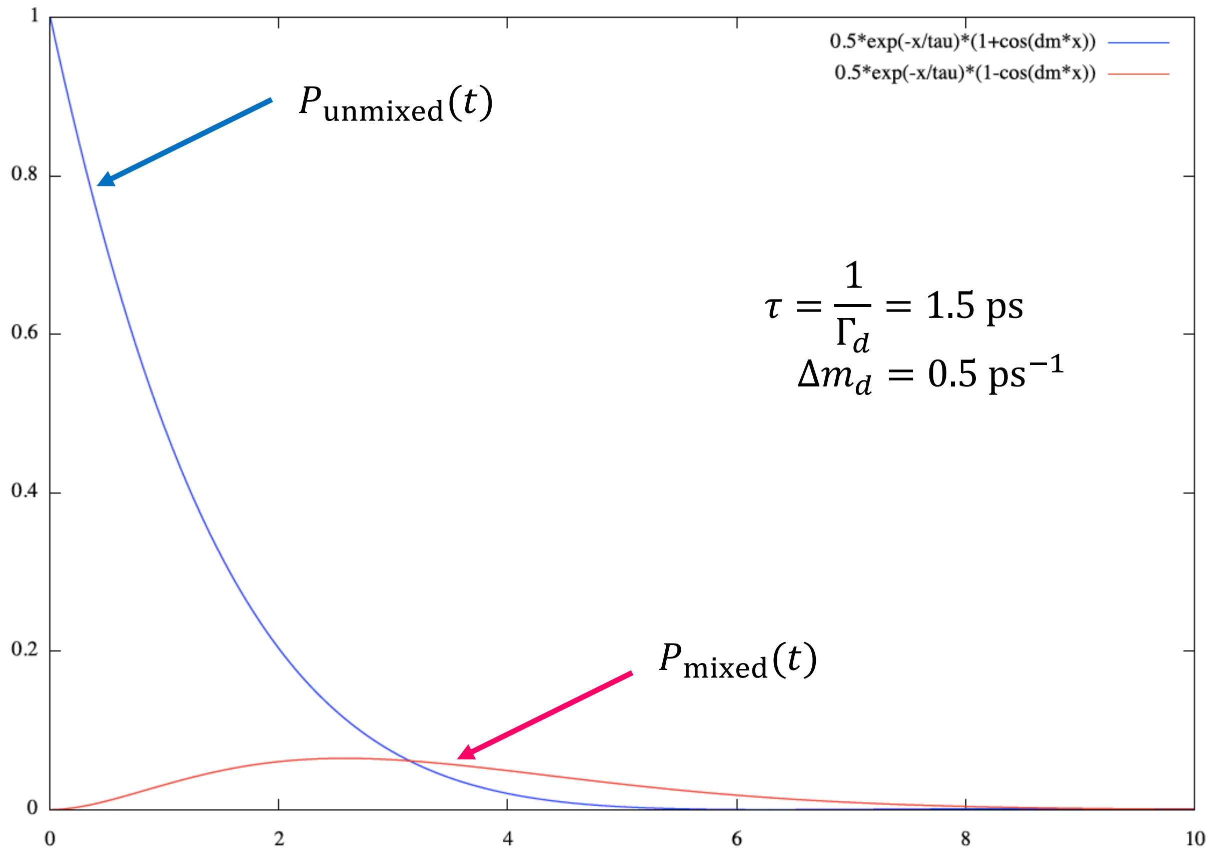
- In a system that the approximations $|q/p| \approx 1$ and $\Delta\Gamma \approx 0$ hold,

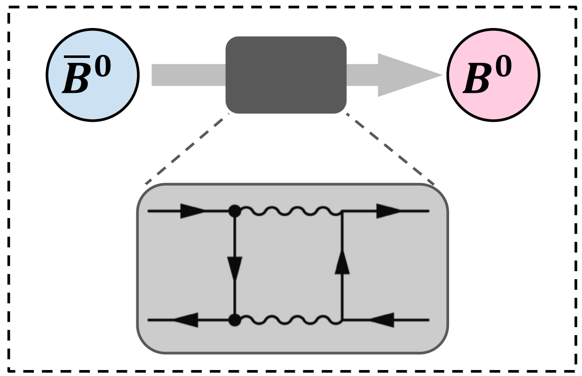
$$P(B^0 \to B^0; t) = \frac{1}{2}\exp(-\Gamma_d t)\,[1 + \cos(\Delta m_d t)]\ \text{ ... unmixed (same eq. for } \bar{B}^0 \to \bar{B}^0)$$

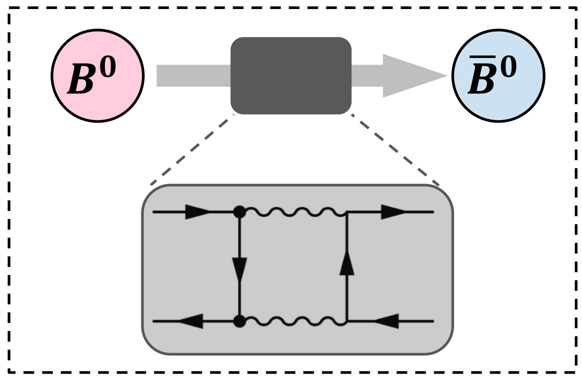
$$P(B^0 \to \bar{B}^0; t) = \frac{1}{2}\exp(-\Gamma_d t)\,[1 - \cos(\Delta m_d t)]\ \text{ ... mixed (same eq. for } \bar{B}^0 \to B^0)$$

Application: B^0-$\bar{B}^0$ Mixing

$$P_{\text{unmixed}}(t) = \frac{1}{2}\exp(-\Gamma_d t)\,[1 + \cos(\Delta m_d t)],\;\; P_{\text{mixed}}(t) = \frac{1}{2}\exp(-\Gamma_d t)\,[1 - \cos(\Delta m_d t)]$$







Again, the B^0-$\bar{B}^0$ mixing is an independent phenomenon of the $B^0\bar{B}^0$ pair creation.

Application: B^0-$\bar{B}^0$ Mixing

$$\int_0^{+\infty} dt \frac{P_{\text{mixed}}(t)}{P_{\text{unmixed}}(t) + P_{\text{mixed}}(t)} = \frac{\Delta m_d^2}{2(\Gamma_d^2 + \Delta m_d^2)} \equiv \chi_d$$

B^0- $\bar{B}^0$ Mixing (1986, 1987)

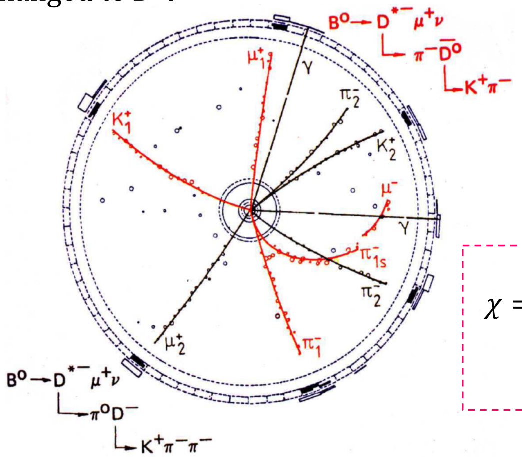
ARGUS Collaboration,
Phys. Lett. **B192**, 245 (1987)

- B^0- $\bar{B}^0$ mixing: $\chi = \frac{N(B^0 \to \bar{B}^0)}{N(B^0 \to B^0) + N(B^0 \to \bar{B}^0)}$. The $\chi = 0$ if no mixing.

$N(B^0 \to B^0)$ ... unmixed $N(B^0 \to \bar{B}^0)$ ... mixed

Evidence for the B^0- $\bar{B}^0$ mixing

$\bar{B}^0$ pairly-produced with B^0 from the e^+-e^- collision by the DORIS II accelerator had changed to B^0.



$$\chi = \frac{N(B^0 \to \bar{B}^0)}{N(B^0 \to B^0) + N(B^0 \to \bar{B}^0)} = 0.17 \pm 0.05$$

$\chi \equiv x^2/2(1+x^2), x \equiv \Delta m_{B^0}/\Gamma_{B^0}$.

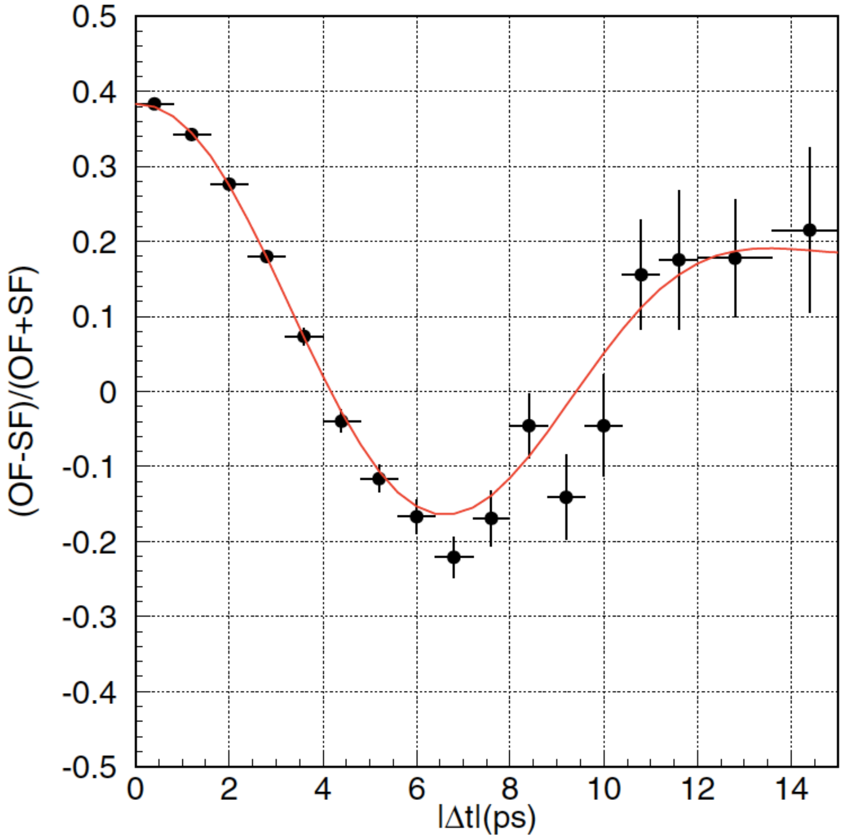
https://argus-fest.desy.de/e301/e305/wsp_arg_new.pdf

$\chi_{B^0}^{\text{PDG2020}} = \mathbf{0.1858 \pm 0.0011}$

The χ_d is sometimes called a time-integrated mixing parameter.

Measurement of Δm_d at Belle

Δm_d results from Belle II will come soon…



Asymmetry $A(t) \equiv \dfrac{P_{\text{unmixed}}(t) - P_{\text{mixed}}(t)}{P_{\text{unmixed}}(t) + P_{\text{mixed}}(t)}$

When the effects from the wrong tagging probability and vertex reconstruction resolution are negligible, $\boldsymbol{A(t) = \cos \Delta m_d t}$.

$$\boldsymbol{\Delta m_d = (0.511 \pm 0.005 \pm 0.006)\ \text{ps}^{-1}}$$

with 152M $B\bar{B}$

Belle Collaboration, Phys. Rev. D **71**, 072003 (2005).

The Δt must be used in the equation above instead of the t. This "misuse" is on purpose to simplify the discussion here. The replacement of the t with the Δt is discussed in the next few pages.

Application: B^0-$\bar{B}^0$ Coherence at Belle (II)

- A critical difference between the B^0-$\bar{B}^0$ mixing in a general setup and that in the Belle (II) setup: **at Belle and Belle II, two B-mesons are pair-produced and they are entangled** (i.e.: the two B mesons take only the $(B^0, \bar{B}^0)$ state until one of them decays).

- The wavefunction of the two-B-meson system $|\text{Belle}(t_1, t_2)\rangle$ at the time of the pair production, $t_1 = 0$ and $t_2 = 0$, is

✓ $$|\text{Belle}(0,0)\rangle = \frac{1}{\sqrt{2}}\left(\left|B_1^0 \bar{B}_2^0\right\rangle - \left|\bar{B}_1{}^0 B_2^0\right\rangle\right)$$
Exchange asymmetric

✗ $$|\text{Belle}(0,0)\rangle = \frac{1}{\sqrt{2}}\left(\left|B_1^0 \bar{B}_2^0\right\rangle + \left|\bar{B}_1{}^0 B_2^0\right\rangle\right)$$
Exchange symmetric

- The general wavefunction to observe one B at $t = t_1$ and the other at $t = t_2$ is

$$|\text{Belle}(t_1, t_2)\rangle = \frac{1}{\sqrt{2}}\left(\left|B_1^0(t_1)\right\rangle\left|\bar{B}_2^0(t_2)\right\rangle - \left|\bar{B}_1^0(t_1)\right\rangle\left|B_2^0(t_2)\right\rangle\right)$$

- By recalling $|B^0(t)\rangle \equiv f_+(t)\left|B^0\right\rangle + \frac{q}{p} f_-(t)\left|\bar{B}^0\right\rangle$, $|\bar{B}^0(t)\rangle \equiv \frac{p}{q} f_-(t)\left|B^0\right\rangle + f_+(t)\left|\bar{B}^0\right\rangle$

$$|\text{Belle}(t_1, t_2)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{\Gamma}{2}(t_1+t_2)} \times$$
$$\left[i \sin\frac{\Delta m_d (t_2 - t_1)}{2}\left(\frac{p}{q}\left|B_1^0 B_2^0\right\rangle - \frac{q}{p}\left|\bar{B}_1^0 \bar{B}_2^0\right\rangle\right) + \cos\frac{\Delta m_d (t_2 - t_1)}{2}\left(\left|B_1^0 \bar{B}_2^0\right\rangle - \left|B_1^0 \bar{B}_2^0\right\rangle\right)\right]$$

Application: CPV in *B* Decays at Belle (II)

- $$|\mathrm{Belle}(t_1,t_2)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{\Gamma}{2}(t_1+t_2)} \times$$
$$\left[i \sin\frac{\Delta m_d (t_2 - t_1)}{2} \left(\frac{p}{q} \left| B_1^0 B_2^0 \right\rangle - \frac{q}{p} \left| \bar{B}_1^0 \bar{B}_2^0 \right\rangle \right) + \cos\frac{\Delta m_d (t_2 - t_1)}{2} \left(\left| B_1^0 \bar{B}_2^0 \right\rangle - \left| B_1^0 \bar{B}_2^0 \right\rangle \right) \right]$$

- Define the $B \to f_{CP}$ decay amplitudes by
$\langle f_{CP}|\mathcal{H}_d|B^0\rangle \equiv \mathcal{A}_{f_{CP}}$, $\langle f_{CP}|\mathcal{H}_d|\bar{B}^0\rangle \equiv \bar{\mathcal{A}}_{f_{CP}}$, and $\lambda_{f_{CP}} \equiv \dfrac{\bar{\mathcal{A}}_{f_{CP}}}{\mathcal{A}_{f_{CP}}} \cdot \dfrac{q}{p}$.

- Define the flavor-specific B-decay amplitudes by

$\langle \ell^+ D^- \nu_\ell|\mathcal{H}_d|B^0\rangle = \bar{\mathcal{A}}_{\ell^+}$, $\langle \ell^+ D^- \nu_\ell|\mathcal{H}_d|\bar{B}^0\rangle = 0$,

$\langle \ell^- D^+ \bar{\nu}_\ell|\mathcal{H}_d|B^0\rangle = 0$, $\langle \ell^- D^+ \bar{\nu}_\ell|\mathcal{H}_d|\bar{B}^0\rangle = \bar{\mathcal{A}}_{\ell^-}$,

Forbidden in the SM.

The same in the SM $\equiv \mathcal{A}_{SL}$.

- Then, the probability to find a signature for the $[B^0\bar{B}^0] \to [(\ell^- D^+ \bar{\nu}_\ell)_1 (f_{CP})_2]$ in the Belle (II) detector at the time t_1 and t_2, respectively, and

  the probability to find a signature for the $[B^0\bar{B}^0] \to [(\ell^+ D^- \nu_\ell)_1 (f_{CP})_2]$ in the Belle (II) detector at the time t_1 and t_2, respectively,

  are computed as the ones on the next page.

Application: CPV in *B* Decays at Belle (II)

- The probability to find a signature for the $[B^0\bar{B}^0] \to [(\ell^- D^+ \bar{\nu}_\ell)_1 (f_{CP})_2]$ in the Belle (II) detector at the time t_1 and t_2, respectively, is:

$$\langle(\ell^- D^+ \bar{\nu}_\ell)_2 (f_{CP})_1|\mathcal{H}_d|\mathrm{Belle}(t_1,t_2)\rangle = \frac{1}{2} e^{-\Gamma(t_1+t_2)} |\mathcal{A}_{SL}|^2 \left|\mathcal{A}_{f_{CP}}\right|^2 \times$$
$$\times \left[\frac{\left(1-\left|\lambda_{f_{CP}}\right|^2\right)+\left(1-\left|\lambda_{f_{CP}}\right|^2\right)}{2} \cos[\Delta m_d(t_2-t_1)] - \mathcal{Im}(\lambda_{f_{CP}}) \sin[\Delta m_d(t_2-t_1)]\right]$$

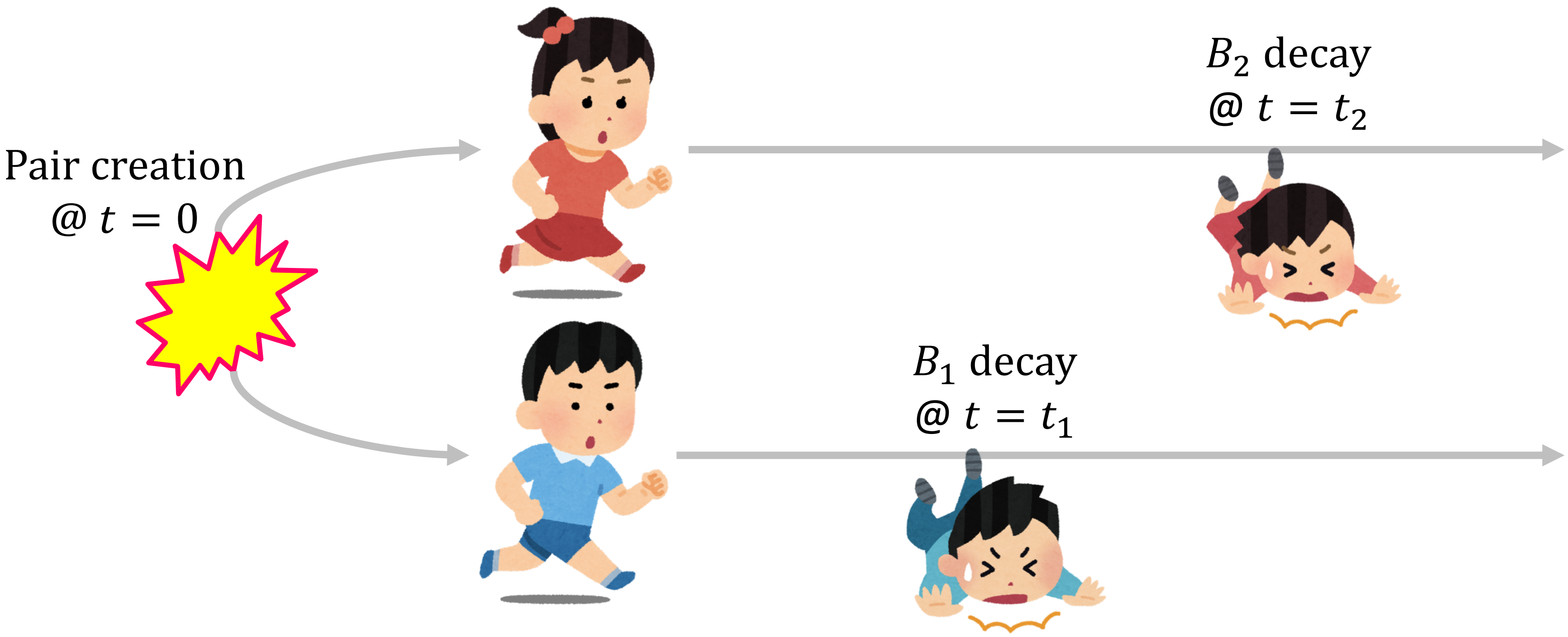
- The probability to find a signature for the $[B^0\bar{B}^0] \to [(\ell^+ D^- \nu_\ell)_1 (f_{CP})_2]$ in the Belle (II) detector at the time t_1 and t_2, respectively, is:

$$\langle(\ell^+ D^- \nu_\ell)_2 (f_{CP})_1|\mathcal{H}_d|\mathrm{Belle}(t_1,t_2)\rangle = \frac{1}{2} e^{-\Gamma(t_1+t_2)} |\mathcal{A}_{SL}|^2 \left|\mathcal{A}_{f_{CP}}\right|^2 \times (p/q)^2$$
$$\times \left[\frac{\left(1+\left|\lambda_{f_{CP}}\right|^2\right)-\left(1-\left|\lambda_{f_{CP}}\right|^2\right)}{2} \cos[\Delta m_d(t_2-t_1)] + \mathcal{Im}(\lambda_{f_{CP}}) \sin[\Delta m_d(t_2-t_1)]\right]$$

- By approximating $\left|\mathcal{A}_{f_{CP}}/\bar{\mathcal{A}}_{f_{CP}}\right| \approx 1$ and $|q/p| \approx 1$,

$$\left\langle(\ell^\pm D\nu)_2 (f_{CP})_1\middle|\mathcal{H}_d\middle|\mathrm{Belle}(t_1,t_2)\right\rangle \propto e^{-\Gamma(t_1+t_2)}\left[1 \pm \mathcal{Im}(\lambda_{f_{CP}}) \sin[\Delta m_d(t_2-t_1)]\right]$$

Application: CPV in B Decays at Belle (II)



- The Belle (II) detector cannot measure the absolute time from the pair-creation of the two B mesons to their decay → absolute time t_1 or t_2 cannot be measured, individually. But it can **measure the time difference $\Delta t \equiv t_2 - t_1$**.

- To account for the ambiguity in t_1 and t_2, apply $\int_0^{+\infty} dt_2 \int_0^{+\infty} dt_1 \ \delta(t_2 - t_1 - \Delta t)$ to $\langle (\ell^\pm D\nu)_2 (f_{CP})_1 | \mathcal{H}_d | \mathrm{Belle}(t_1, t_2) \rangle$.

Application: CPV in *B* Decays at Belle (II)

- $$\int_0^{+\infty} dt_2 \int_0^{+\infty} dt_t \;\delta(t_2 - t_1 - \Delta t)\left\langle (\ell^{\pm} D\nu)_2 (f_{CP})_1 \middle| \mathcal{H}_d \middle| \mathrm{Belle}(t_1, t_2)\right\rangle$$

  $$\propto e^{-\Gamma|\Delta t|}\left(1 \pm \mathcal{I}m\left(\lambda_{f_{CP}}\right)\sin \Delta m_d \Delta t\right).$$

  By replacing the Γ with τ_{B^0} and taking the normalization,

$$\boldsymbol{P(t;\ell^{\pm}) = \frac{1}{2\tau_{B^0}} e^{-\frac{|\Delta t|}{\tau_{B^0}}}\left(1 \pm \mathcal{I}m\left(\lambda_{f_{CP}}\right)\sin \Delta m_d \Delta t\right)}$$

- By applying a similar discussion to the B^0-$\bar{B}^0$ mixing, one obtains

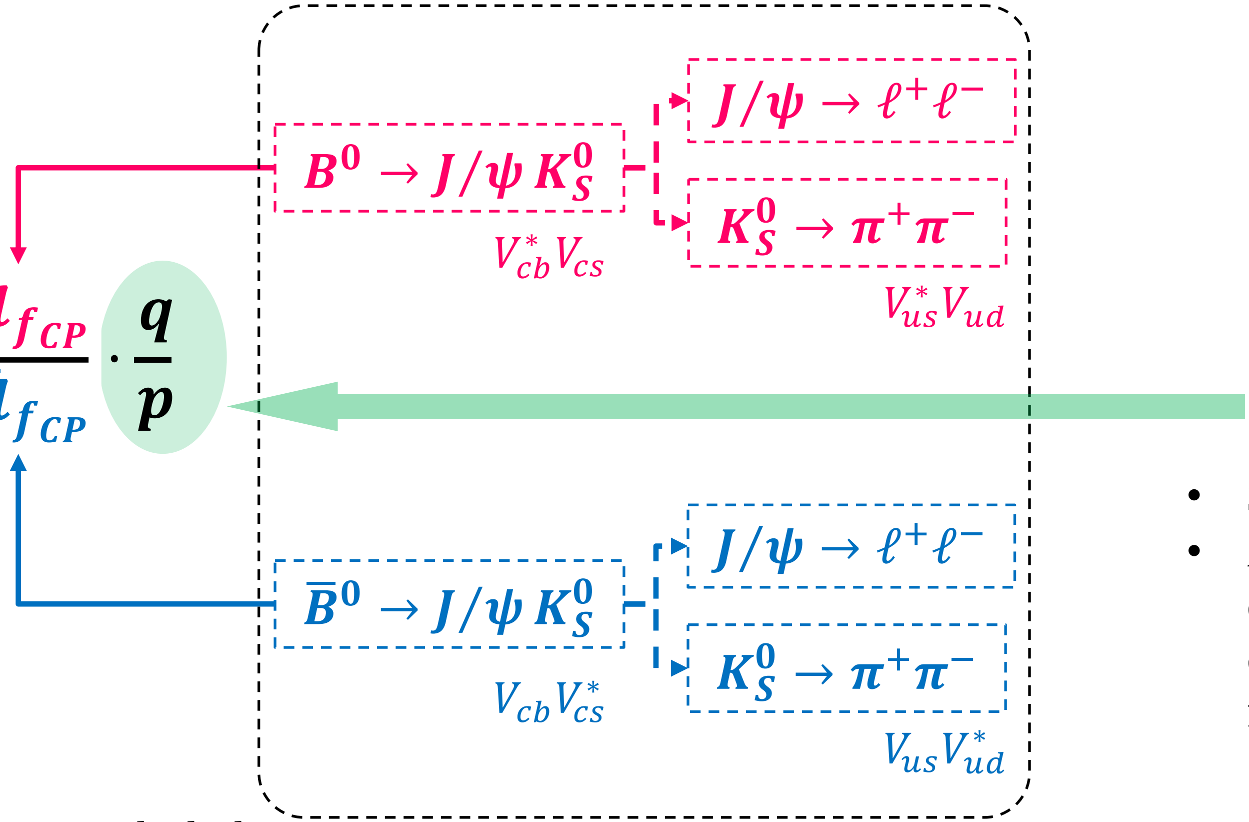
$$P(t;\ell^{\pm}) = \frac{1}{2\tau_{B^0}} e^{-\frac{|\Delta t|}{\tau_{B^0}}}\left(1 \pm \cos \Delta m_d \Delta t\right)$$

$+$ for unmixed
$-$ for mixed

The Last Piece, $\lambda_{f_{CP}}$

- Assume we use the golden mode for the test of the Kobayashi-Maskawa theory, where $B^0 \to J/\psi\, K_S^0$ and $\bar{B}^0 \to J/\psi\, K_S^0$.

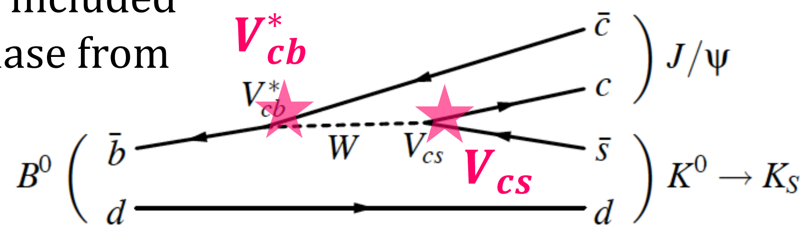
$$\lambda_{f_{CP}} \equiv \frac{\mathcal{A}_{f_{CP}}}{\bar{\mathcal{A}}_{f_{CP}}} \cdot \frac{q}{p}$$

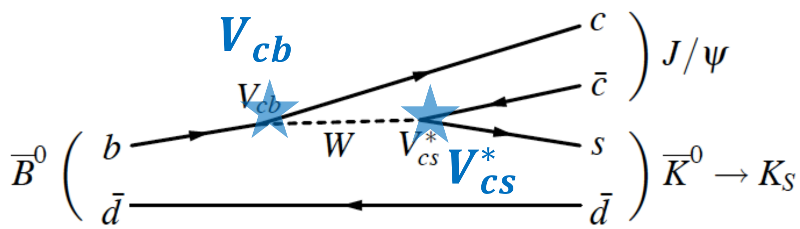


$$\frac{q}{p} \approx \frac{V_{tb}^* V_{td}}{V_{tb} V_{td}^*}$$

- Skipped derivation.
- K^0-$\bar{K}^0$ mixing is omitted for its small contribution to the B-meson system.

- No V_{td} or V_{ub} included → no CKM phase from the decay.





Application: CPV in *B* Decays at Belle (II)

- For $B^0(\bar{B}^0) \to J/\psi\, K_S^0$, $\lambda_{J/\psi K_S^0} \equiv \frac{\mathcal{A}_{f_{CP}}}{\bar{\mathcal{A}}_{f_{CP}}} \cdot \frac{q}{p} = \xi_{J/\psi K_S^0} \frac{V_{cb}^* V_{cs} V_{us}^* V_{ud}}{V_{cb} V_{cs}^* V_{us} V_{ud}^*} \cdot \frac{V_{tb}^* V_{td}}{V_{tb} V_{td}^*}$

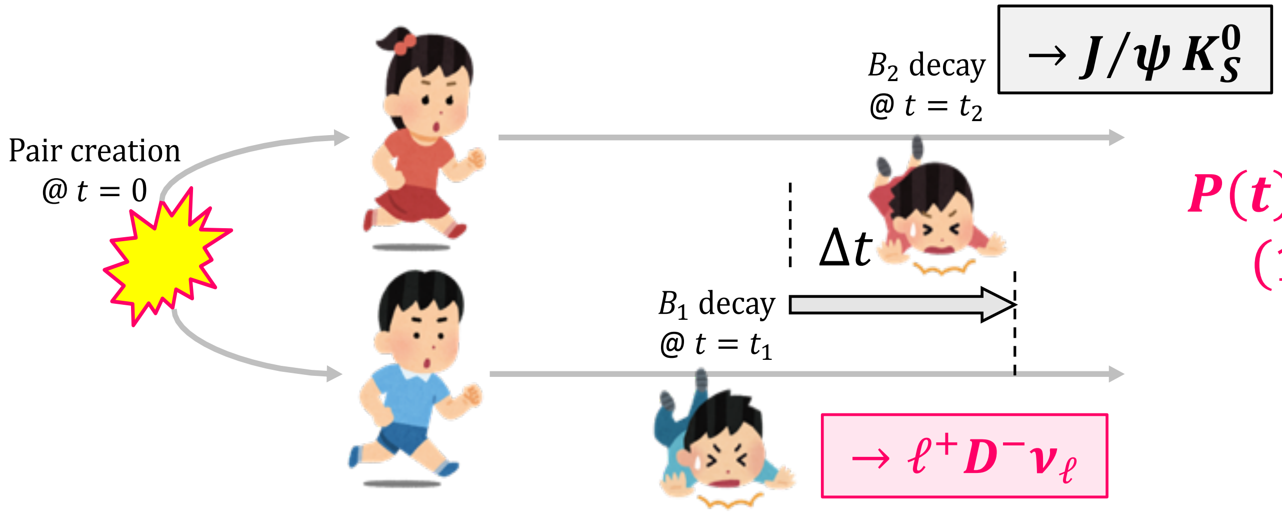
  Since only V_{td} and V_{ub} are complex, $\lambda_{J/\psi K_S^0} = \xi_{J/\psi K_S^0} \cdot e^{-2i\phi_1}$.

  $\mathcal{Im}\left(\lambda_{J/\psi K_S^0}\right) = -\xi_{J/\psi K_S^0} \sin 2\phi_1 = \sin 2\phi_1$.

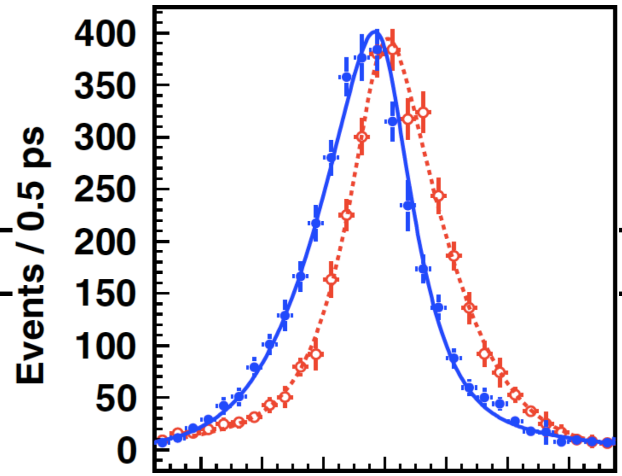
$$\boldsymbol{P(t;\ell^\pm) = \frac{1}{2\tau_{B^0}} e^{-\frac{|\Delta t|}{\tau_{B^0}}} (1 \pm \sin 2\phi_1 \sin \Delta m_d \Delta t)}$$

- $\mathcal{Im}(\lambda_{f_{CP}})$ depends on the $\mathcal{A}_{f_{CP}}$ and $\bar{\mathcal{A}}_{f_{CP}}$, which are determined by the chosen $B^0(\bar{B}^0) \to f_{CP}$ mode. For example, if one chooses $B^0(\bar{B}^0) \to \pi^+\pi^-$, he/she obtains the $P(t;\ell^\pm)$ equation with $\sin 2\phi_2$.

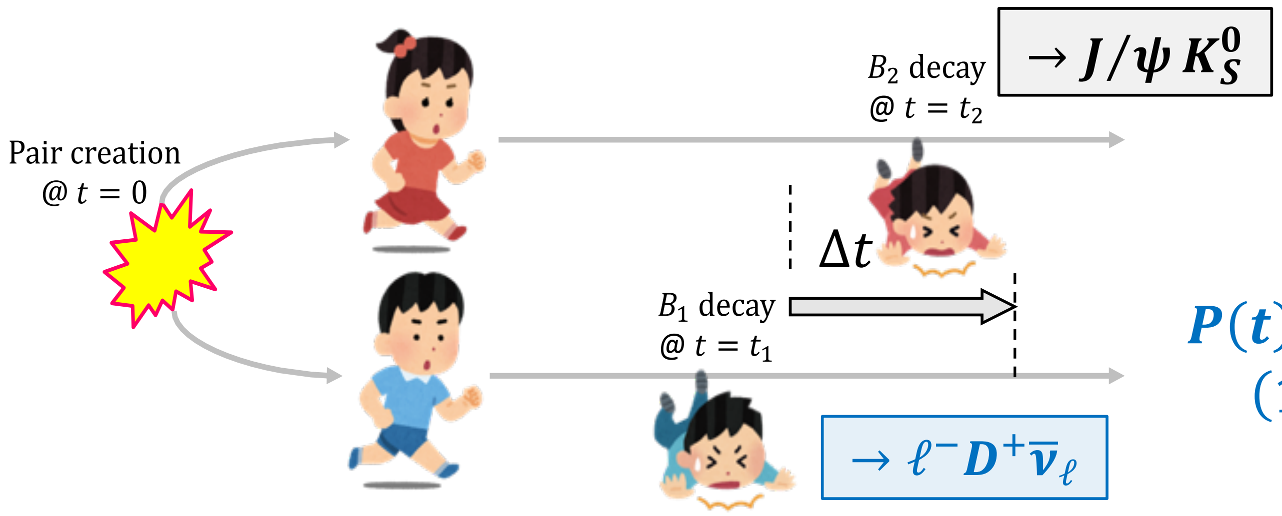
Hitchhikers' Summary



$$P(t) \propto e^{-|\Delta t|/\tau_{B^0}} \times (1 + \sin 2\phi_1 \sin \Delta m_d \Delta t)$$



Belle Collaboration, Phys. Rev. Lett. **108**, 171802 (2012).



$$P(t) \propto e^{-|\Delta t|/\tau_{B^0}} \times (1 - \sin 2\phi_1 \sin \Delta m_d \Delta t)$$

Three Types of *CP* Violation

1. Direct *CP* violation

Matter-antimatter asymmetry when $\mathcal{A}_{f_{CP}} \neq \bar{\mathcal{A}}_{f_{CP}}$. The decay final state is not necessary a *CP* eigenstate, which means $\mathcal{A}_f \neq \bar{\mathcal{A}}_{\bar{f}}$.

2. Indirect *CP* violation

Matter-antimatter asymmetry when $|q/p| \neq 1$ while we usually assume $|q/p| \neq 1$ in the B-meson system.

Measure with the time-dependent analysis method

3. Mixing induced *CP* violation

Matter-antimatter asymmetry in the interference between with and without mixing (i.e.: $B^0 \to \bar{B}^0, \bar{B}^0 \to B^0$ and $B^0 \to B^0, \bar{B}^0 \to \bar{B}^0$, respectively).

That’s All.