Form Factor-ology

We wrote down above expression of the hadronic matrix elements in terms of form factors. These are (Lorentz invariant) amplitudes; a unique one for each allowed combination of momenta, polarisation or spinors, that represent the external states (roughly). Eg we had

Qn: Where does this come from? →

$$\langle D^*_\lambda(p')|\bar c\gamma^\mu b|\bar B(p)\rangle = ig(q^2)\,\varepsilon^{\mu\nu\alpha\beta}\,\varepsilon^*_{\lambda\nu}\,p'_\alpha\, p_\beta$$

$$\langle D^*_\lambda(p')|\bar c\gamma^\mu\gamma^5 b|\bar B(p)\rangle = f(q^2)\,\varepsilon^{*\mu}_\lambda + a_+(q^2)\,\varepsilon^*_\lambda\cdot p\,(p+p')^\mu + a_-(q^2)\,\varepsilon^*_\lambda\cdot p\, q^\mu \qquad (1)$$

The B and D^* are states of definite $J^P = 0^-$ and 1^-, respectively. The operators decompose into pieces with definite J^P as:

$$V: \quad \bar c\gamma^\mu b \sim 0^+ \oplus 1^-$$
$$A: \quad \bar c\gamma^\mu\gamma^5 b \sim 0^- \oplus 1^+ \qquad (2)$$

To deduce the structure in (1), can think about the crossed process $\langle D^* B|\bar c\Gamma b|0\rangle$ in which D^*B has spin parity $S^{P'} = 1^-\otimes 0^- = 1^+$. We can tensor this with representations of partial waves $L=0,1,2,\ldots$ with parity $P=(-1)^L$.

Idea: • Write down allowed partial waves (definite J^P) of the D^*B state.

• Each one that matches onto ~~expression~~ an operator in (2) generates a unique form factor.

So, allowed partial waves, $S^{P'}$ L^P

$$\begin{aligned}
J^P(L=0) &= 1^+\otimes 0^+ = 1^+ \\
J^P(L=1) &= 1^+\otimes 1^- = 0^-\oplus 1^-\oplus 2^- \\
J^P(L=2) &= 1^+\otimes 2^+ = 1^+\oplus 2^+\oplus 3^+
\end{aligned} \qquad (3)$$

For $L\geq 3$ no more $1^\pm$ or $0^\pm$ that can match onto (2).

So we find

Operator	FF(J^P)	
Vector	1^-	g
Axialvector	0^- 1^+ 1^+	

$0^- \to P_1$ (couples to $\frac{q^\mu q^\nu}{q^2}$), $1^+ \to f$, $1^+ \to F_1$

Only $J^P = 1^-$ tensor involves Levi Civita (in p' rest frame $\epsilon^{\mu\nu\rho\sigma} \varepsilon_\nu p'_\rho p_\sigma$

$\Rightarrow \epsilon^{ijok} \varepsilon_j p'_o p_k$

$\to -\underbrace{\varepsilon_p}_{\text{spin } 1}^{ijok} \varepsilon_j p'_o p_k$ under P)

Classification of the FFs in terms of J^P allows us to deduce analytic properties of the form factors. First, in general

$$F = F(q^2; \text{parameters}) \qquad — (4)$$

↳ $\underbrace{M_B, M_{D^*}, m_b, m_c, \text{-----}}_{\text{FF parametrization}}$

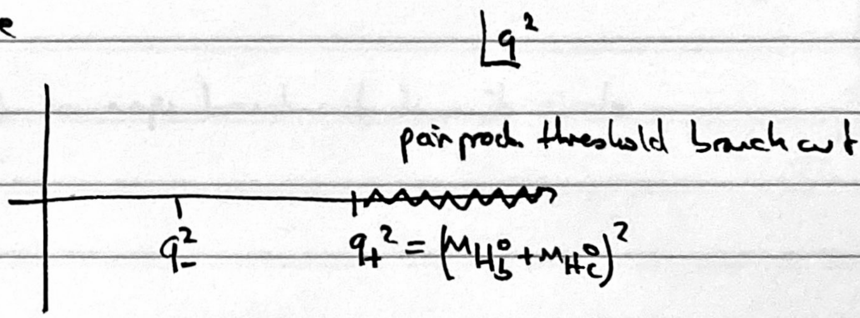
or $w = \dfrac{M_B^2 + M_{D^*}^2 - q^2}{2 M_B M_{D^*}}$ — (5b)

Recall: $m_\ell^2 \leq q^2 \leq q_-^2 = (M_B - M_{D^*})^2$

$\dfrac{1+r^2-r_\ell^2}{2r} \geq w \geq 1$ — (5a)

← "zero recoil"
D^* at rest in B frame.

For the crossed process $\langle D^* B | \mathcal{O}_{J^P} | 0 \rangle$, in the complex q^2 plane the amplitude has structure



Here $H_b^0 H_c^0$ is the lightest hadron pair coupling to $\mathcal{O}_{J^P}$

1^-	BD
$0^-\ 1^+$	$B^* D$

Despite this, common to just take $q_+^2 = (M_B + M_{D^*})^2$ for all J^P. On the branch cut, $q^2 > q_+^2$ has a special relation to QCD correlators computable in perturbative QCD. In particular, once or twice "subtracted" correlators

computable in pert. QCD $\dfrac{\partial^n \Pi_J}{\partial(q^2)^n}$

$$\chi_J = \frac{1}{\pi}\int_0^\infty \frac{dt}{(t-q^2)^2} \underbrace{\frac{1}{2}\sum_X (2\pi)^4 \delta^4(q-p_X)\left|\langle 0|J|X\rangle\right|^2}_{\text{Im of 2pt function } \Pi_J}$$

$$\geq \frac{1}{\pi}\int_0^\infty \frac{dt}{(t-q^2)} \frac{1}{2} (2\pi)^4 \delta^4(q-p_{BD^*})\left|\langle 0|J|BD^*\rangle\right|^2 \quad -⑥$$

unitarity

applying crossing symmetry to $\langle 0|J|X\rangle$

$$\int_{q^2>q_+^2} dq^2 \sum_i \left|\phi^J(q^2)\, F_i^J(q^2)\right| \leq \pi \chi_J \quad -⑦$$

various FFs

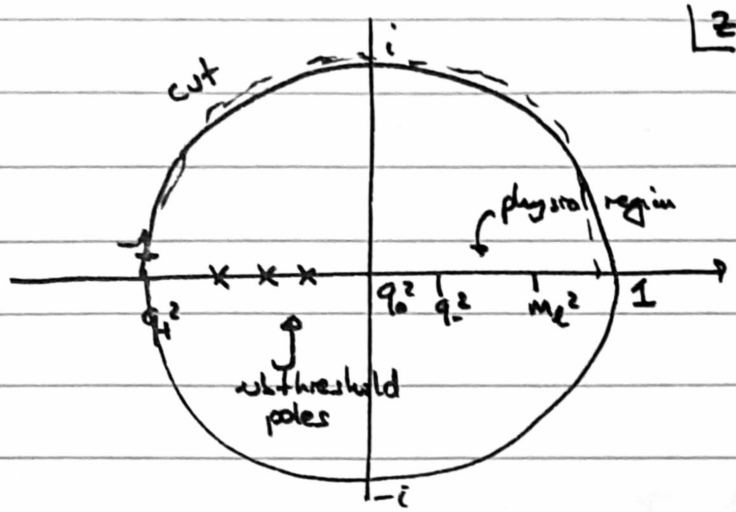
"weight" or "outer" function from $\langle D^*|J|B\rangle$ and $\frac{1}{(t-q^2)^n}$ factor.

Can exploit ⑦ using a conformal transformation

$$z(q^2, q_0^2) = \frac{\sqrt{q_+^2-q^2} - \sqrt{q_+^2-q_0^2}}{\sqrt{q_+^2-q^2} + \sqrt{q_+^2-q_0^2}} \qquad q^2 \leq q_0^2 < q_+^2 \quad -⑧$$

arbitrary choice

Note $|z(q^2 > q_+^2, q_0^2)| = 1$ so maps branch cut to unit circle



Typical to choose $q_0^2 = q_-^2$

Then $z(q_-^2, q_-^2) = 0$

at $w = 1$.

There is another choice

$$q_0^2 = q_+^2\left(1 - \sqrt{1 - q_-^2/q_+^2}\right) \equiv q_{opt}^2$$

that minimizes $|z|$ on physical range. Often $z_* = z(q^2, q_{opt}^2)$

Under ⑧, ⑦ becomes

$$\oint_{|z|=1} \frac{dz}{2\pi i z} \sum_i \left| \phi_i^J(z)\, F_i^J(z) \right|^2 \leq 1 \qquad -⑨$$

Complication: On $q_-^2 < q^2 < q_+^2$ subthreshold region, there can be poles from bound states: "subthreshold resonances", that render integrand non-analytic. But, define "Blaschke Factor"

$$P^J(z) = \prod_\alpha \left(\frac{z - z_\alpha^J}{1 - z\, z_\alpha^J} \right) \quad \text{at each } z_\alpha^J \text{ pole.} \qquad -⑩$$

data and/or LQCD

These have property $|P^J(z)| = 1$ on $|z| = 1$ so do not affect boundary integral. But, they cancel the poles. Hence $P^J(z)\, \phi_i^J(z)\, F_i^J(z)$ is analytic: Can be written as $\sum_n a_n^{Ji} z^n$, ie

$$F_i^J(z) = \frac{1}{P^J(z)\, \phi_i^J(z)} \sum_n a_n^{Ji} z^n \qquad -⑪$$

while ⑨ implies

$$\sum_{i,n} \left|a_n^{Ji}\right|^2 \leq 1 \qquad -(12)$$

Specifically, for $B \to D^*$

$$g(z) = \frac{1}{P^{1^-}(z)\,\phi_g(z)} \sum_n a_n^g z^n \,, \qquad \underbrace{\sum_n \big|\overbrace{a_n^g}^{\text{"}a_n\text{"}}\big|^2 \leq 1}_{\text{weak unitarity bound}}$$

"BGL"
hep-ph/9705252

$$f(z) = \frac{1}{P^{1^+}(z)\,\phi_f(z)} \sum_n a_n^f z^n \qquad \sum_n |\underbrace{a_n^f}_{\text{"}b_n\text{"}}|^2 + |\underbrace{a_n^{F_1}}_{\text{"}c_n\text{"}}|^2 \leq 1 \qquad -(13)$$

$$F_1(z) = \frac{1}{P^{1^+}(z)\,\phi_{F_1}(z)} \sum_n a_n^{F_1} z^n$$

and note also from ㉓ in part 1: $F_1(0) = m_B(1-r)\, f(0)$ so

$$a_1^{F_1} \big/ \phi_{F_1}(0) = m_B(1-r)\; a_1^f \big/ \phi_f(0) \qquad -(14).$$

Note there is no prediction for P_1, which has its own expansion

$$P_1(z) = \frac{1}{P^{0^-}(z)\,\phi_{P_1}(z)} \sum_n a_n^{P_1} z^n \,, \qquad \sum_n |\underbrace{a_n^{P_1}}_{\text{"}d_n\text{"}}|^2 \leq 1 \qquad -(15)$$

The BGL parameters are expressed wrt the matrix element. Common to define

$$\tilde{a}_n = \eta_{EW} V_{cb}\, a_n \qquad -(16)$$

so care required with normalization.

How do we develop predictions for FFs like P_1 from $\bar{B} \to D^{(*)} \ell \nu$, $m_\ell / m_{B,D^*} << 1$ & data, or for new physics operators beyond standard model? I.e. for

$$J_S = \bar{c} b, \quad J_P = \bar{c}\gamma^5 b, \quad J_V = \bar{c}\gamma^\mu b, \quad J_A = \bar{c}\gamma^\mu\gamma^5 b, \quad J_T = \bar{c}\sigma^{\mu\nu} b \qquad -(17)$$

One way is a quark model, that approximates knowledge guess of hadron wavefunctions.
A (much) more model independent approach uses Heavy Quark Effective Theory (HQET)

Basic insight: quark propagator

$$D_Q(p) = \frac{1}{\not{p}_Q - m_Q}, \quad p_Q = m_Q v + k$$

$k \sim \Lambda_{QCD}$; v: HQ velocity

$$= \frac{m_Q(\not{v}+1) + \not{k}}{2 m_Q v\cdot k + k^2}$$

$$\simeq \frac{1+\not{v}}{2}\frac{1}{v\cdot k} + \mathcal{O}\left(\frac{\Lambda_{QCD}}{m_Q}\right) \qquad -(18)$$

$\frac{1}{v\cdot k}$: suggests a 2pt function $\sim v\cdot D$

$\frac{1+\not{v}}{2}$: projectors

$$\Pi_\pm = \frac{1 \pm \not{v}}{2} \quad (\Pi_+\Pi_- = 0, \ \Pi_\pm^2 = \Pi_\pm)$$

so write HQ fields

$$Q_\pm^v(x) = e^{i m_Q v\cdot x}\, \Pi_\pm Q(x) \qquad -(19)$$

$e^{i m_Q v\cdot x}$: "mass subtraction"; $Q(x)$: quark field

Can show

$$\mathcal{L}_{QCD} = \bar{Q}(i\not{D} - m_Q) Q$$
$$\equiv \bar{Q}_+^v\, i v\cdot D\, Q_+^v + \bar{Q}_+^v\, i\not{D}_\perp Q_-^v + \bar{Q}_-^v\, i\not{D}_\perp Q_+^v - \bar{Q}_-^v (i v\cdot D + 2m_Q) Q_-^v \qquad -(20)$$

$D_\perp$: $D_\mu - (v\cdot D) v_\mu$

Q_+^v: massless HQ field, yields $\frac{\Pi_+}{v\cdot k}$ 2pt. fn.

Q_-^v: double heavy HQ

Idea: Integrate out Q_-^v, yielding an effective theory order by order in $\frac{iv\cdot D}{2M_Q}$ wrt Q_+^v

$$\mathcal{L}_{HQET} = \underbrace{\bar{Q}_+^v\, iv\cdot D\, Q_+^v}_{\mathcal{L}_0} + \bar{Q}_+^v\, i\not{D}_\perp \frac{1}{iv\cdot D + 2M_Q}\, i\not{D}_\perp Q_+^v$$

$$\simeq \bar{Q}_+^v\, iv\cdot D\, Q_+^v - \frac{1}{2M_Q}\bar{Q}_+^v\underbrace{\left[D^2 + a_Q(\mu)\frac{g}{2}\sigma_{\mu\nu}G^{\mu\nu}\right]}_{\mathcal{L}_1}Q_+^v \quad \text{—(21)}$$

$+ \ldots$

The zeroth order, free, term has a HQ spin symmetry, violated by higher terms

Note the velocity is just a label. Common (and convenient) to choose

$$v = p_H/M_H \text{, the } \underline{\text{hadron}} \text{ velocity} \quad \text{—(22)}$$

Can use this to show

$$M_H = M_Q + \underbrace{\bar{\Lambda}}_{\substack{\text{energy of} \\ \text{light dofs in HQ limit}}} - \frac{\lambda_1 + d_H\lambda_2}{2M_Q} \quad \text{—(23)}$$

higher order hadron mass parameters

and, a hadronic (QCD) matrix element

HQET current — HQET states

$$\langle H_c|\bar{c}\Gamma b|H_b\rangle \simeq \langle H_c^{v'}|\bar{c}_+^{v'}\Gamma b_+^v|H_b^v\rangle$$

$$+ \frac{1}{2M_c}\langle H_c^{v'}|\,\underbrace{\bar{c}_+^{v'} J'_+}_{\text{current correction}}\quad \underbrace{T\mathcal{L}_1(x')\,\bar{c}_+^v(x')\Gamma b}_{\text{Lagrangian}}|H_b^v\rangle$$

$$+ \frac{1}{2M_b}(\ldots) \quad \text{—(24)}$$

$+ \ldots$

Isgur Wise function →

Each term on RHS of (24) can be expressed in terms of small set of unknown hadronic functions × calculable traces. At LO

any Γ!

spacetime rep^n $H(v) = \Pi_+[\not{V} - \gamma^5 P]$ for $D^{(*)}$.

$$\langle H_c^{v'}|\bar{c}_+^{v'}\Gamma b_+^v|H_b^v\rangle = \xi(w)\,\mathrm{Tr}\left[H_c(v')\,\Gamma\, H_b(v)\right]$$

obeys $\xi(1) = 1$

... and much much more!