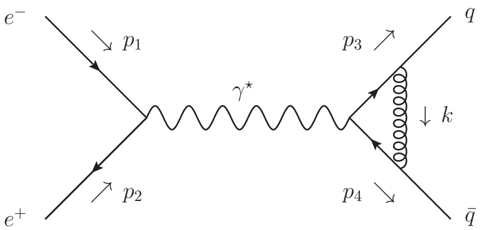
Lecture III

- Virtual corrections
- Dimensional regularization
- Renormalization



$$\mathcal{M}^{(1)}_{q\bar{q}} = \frac{e^2 q_q}{s}\left(\overline{v}(p_2)\gamma_\mu u(p_1)\right)\left(\overline{u}(p_3)\Gamma_\mu v(p_4)\right)$$

$$\overline{u}(p_3)\Gamma_\mu v(p_4) = C_F g_s^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\overline{u}(p_3)\gamma^\alpha \frac{\not{k}+\not{p}_3}{(p_3+k)^2}\gamma^\nu \frac{\not{k}-\not{p}_4}{(p_4-k)^2}\gamma_\alpha v(p_4)$$


DIVERGENCES, DIVERGENCES
LIGHTYEAR
DIVERGENCES EVERYWHERE
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UV Divergences

- Quark propagators in the limit $\boldsymbol{k \gg p_3, p_4}$ (Ultraviolet):

$$\frac{\not{k} + \not{p}_3}{k^2 + 2p_3 \cdot k} \xrightarrow{k \to \infty} \frac{1}{|k|} \qquad \frac{\not{k} - \not{p}_4}{k^2 - 2p_4 \cdot k} \xrightarrow{k \to \infty} \frac{1}{|k|}$$

- The loop integration is divergent

$$\overline{u}(p_3)\Gamma_\mu v(p_4) \stackrel{k\to\infty}{\simeq} \int |k|^3 d|k| d\Omega \frac{1}{|k|^2} \frac{1}{|k|} \frac{1}{|k|} \simeq \int \frac{d|k|}{|k|} \simeq \lim_{|k|\to\infty} \log|k|$$

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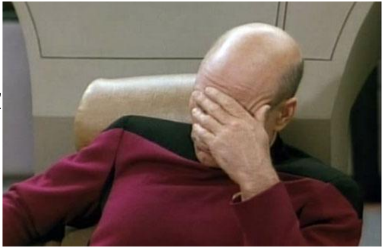

- Quark propagators in the limit $k \ll p_3, p_4$ (Infrared):

$$\frac{\not{k} + \not{p}_3}{k^2 + 2p_3 \cdot k} \xrightarrow{k \to 0} \frac{1}{|k|} \qquad \frac{\not{k} - \not{p}_4}{k^2 - 2p_4 \cdot k} \xrightarrow{k \to 0} \frac{1}{|k|}$$

- The loop integration is divergent

$$\overline{u}(p_3)\Gamma_\mu v(p_4) \overset{k \to 0}{\simeq} \int |k|^3 d|k| d\Omega \frac{1}{|k|^2} \frac{1}{|k|} \frac{1}{|k|} \simeq \int \frac{d|k|}{|k|} \simeq \lim_{|k| \to 0} \log |k|$$

IR Divergences

- Quark propagators in the limit $k \ll p_3, p_4$ (Infrared):

$$\frac{\not{k} + \not{p}_3}{k^2 + 2p_3 \cdot k} \xrightarrow{k \to 0} \frac{1}{|k|} \qquad\qquad \frac{\not{k} - \not{p}_4}{k^2 - 2p_4 \cdot k} \xrightarrow{k \to 0} \frac{1}{|k|}$$

- The loop integration is divergent

$$\overline{u}(p_3)\Gamma_\mu v(p_4) \overset{k \to 0}{\simeq} \int |k|^3 d|k| d\Omega \frac{1}{|k|^2} \frac{1}{|k|} \frac{1}{|k|} \simeq$$

IR Divergences (with gluon mass)

- Quark propagators in the limit $\boldsymbol{k} \ll \boldsymbol{p_3}, \boldsymbol{p_4}$ (Infrared):

$$\frac{\not{k} + \not{p}_3}{k^2 + 2p_3 \cdot k} \xrightarrow{k \to 0} \frac{1}{|k|} \qquad \frac{\not{k} - \not{p}_4}{k^2 - 2p_4 \cdot k} \xrightarrow{k \to 0} \frac{1}{|k|} \qquad \frac{1}{k^2 - \lambda^2} \xrightarrow{k \to 0} \frac{1}{\lambda^2}$$

- The loop integration is regular

$$\overline{u}(p_3)\Gamma_\mu v(p_4) \overset{k \to 0}{\simeq} \int |k|^3 d|k| d\Omega \frac{1}{\lambda^2} \frac{1}{|k|} \frac{1}{|k|} \simeq \int |k| \, d|k| \simeq \lim_{|k| \to 0} |k|$$

Dimensional regularization

- Loop integrals are **regulated** by changing the number of space-time dimensions.
- We evaluate the integrals for a generic value of $d < 4$:

$$\overline{u}(p_3)\Gamma_\mu v(p_4) = C_F g_s^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \overline{u}(p_3)\gamma^\alpha \frac{\not{k} + \not{p}_3}{(p_3+k)^2}\gamma^\nu \frac{\not{k} - \not{p}_4}{(p_4-k)^2}\gamma_\alpha v(p_4)$$

$$\overset{k\to 0}{\simeq} \int |k|^{d-1} d|k| d\Omega \frac{1}{|k|^2}\frac{1}{|k|}\frac{1}{|k|} \simeq \int \frac{d|k|}{|k|^{5-d}}$$

- We substitute $d = 4 - 2\epsilon$ and we assign to the original integral the result obtained from a **Laurent expansion around $\epsilon = 0$**


MAY THE 4-2EPSILON
BE WITH YOU
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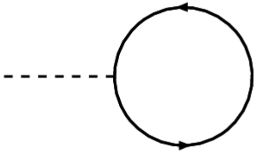
Example: tadpole



$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + \mathrm{i}\delta}$$

- The integral is UV divergent!

- Calculate instead the same integral in d dimensions:

$$I(m^2, d) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2 + \mathrm{i}\delta}$$

- Normalization

$$I(m^2,d) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + \mathrm{i}\delta} \overset{k\to km}{=} (m^2)^{d/2-1} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - 1 + \mathrm{i}\delta}$$

- Wick rotation: convert $k^2 = k_0^2 - (k_1^2 + \cdots + k_{d-1}^2)$ to Euclidean $k_E^2 = k_1^2 + \cdots + k_d^2$!

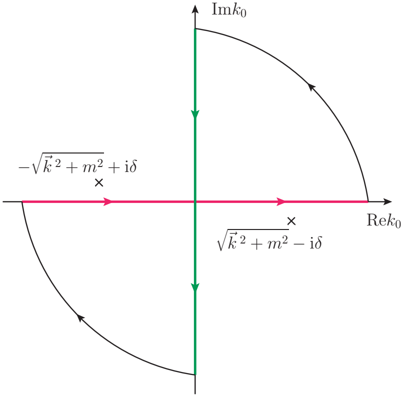
$$I(m^2,d) = \frac{(m^2)^{d/2-1}}{(2\pi)^d} \int_{-\infty}^{+\infty} dk_0 \int d^{d-1}k \, \frac{1}{k^2 - 1 + \mathrm{i}\delta}$$

- There are singularities at

$$k_0^2 - \vec{k}^{\,2} - 1 + \mathrm{i}\delta = 0 \leftrightarrow k_0 = \sqrt{\vec{k}^{\,2} + m^2} - \mathrm{i}\delta \text{ and } k_0 = -\sqrt{\vec{k}^{\,2} + m^2} + \mathrm{i}\delta$$

- Use Cauchy theorem
- Define $k_0 = \mathrm{i}k_d$

$$k^2 = k^2 - \vec{k}^{\,2} = -k_d^2 - \vec{k}^{\,2} = -(k_1^2 + \cdots + k_d^2) = -k_E^2$$



$$\begin{aligned}
I(m^2,d) &= \frac{(m^2)^{d/2-1}}{(2\pi)^d}\int_{-\infty}^{+\infty} dk_0 \int d^{d-1}k\,\frac{1}{k^2-1+\mathrm{i}\delta} \\
\text{Wick:}\quad &= \frac{(m^2)^{d/2-1}}{(2\pi)^d}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} dk_0 \int d^{d-1}k\,\frac{1}{k^2-1} \\
k_0 = \mathrm{i}k_d:\quad &= \mathrm{i}\frac{(m^2)^{d/2-1}}{(2\pi)^d}\int_{-\infty}^{+\infty} dk_d \int d^{d-1}k\,\frac{1}{-k_E^2-1} \\
&= -\mathrm{i}\frac{(m^2)^{d/2-1}}{(2\pi)^d}\int d^d k_E\,\frac{1}{k_E^2+1}
\end{aligned}$$

Example: tadpole

- Spherical coordinates in d dimensions

$$I(m^2, d) = -\mathrm{i}\frac{(m^2)^{d/2-1}}{(2\pi)^d} \underbrace{\int d\Omega_d}_{\text{solid angle}} \underbrace{\int_0^\infty r^{d-1} dr \, \frac{1}{r^2+1}}_{\text{radial integration}} \overset{t=r^2}{=} -\mathrm{i}\frac{(m^2)^{d/2-1}}{(2\pi)^d} \underbrace{\int d\Omega_d}_{\text{solid angle}} \underbrace{\int_0^\infty \frac{dt}{2} \frac{t^{d/2-1}}{t+1}}_{\text{radial integration}}$$

- Solid angle in d dimensions

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \text{ with } \Omega_2 = 2\pi, \Omega_3 = 4\pi, \ldots$$

- Use definition of Euler's beta funtion

$$B(a, b) = \int_0^\infty dt \, \frac{t^{a-1}}{(1+t)^{a+b}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Example: tadpole

$$I(m^2,d) = -\mathrm{i}\frac{(m^2)^{d/2-1}}{(2\pi)^d}\int d\Omega_d \int_0^\infty \frac{dt}{2}\frac{t^{d/2-1}}{t+1}$$

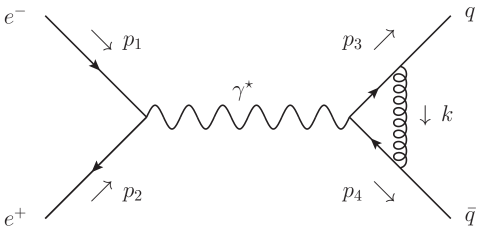
$$= -\mathrm{i}\frac{(m^2)^{d/2-1}}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{2}\frac{\Gamma(d/2)\Gamma(1-d/2)}{\Gamma(1)}$$

$$\overset{d=4-2\epsilon}{=} -\frac{\mathrm{i}}{(4\pi)^{2-\epsilon}}(m^2)^{1-\epsilon}\Gamma(\epsilon-1)$$

- The result expanded in the limit $\epsilon \to 0$

$$I(m^2,\epsilon) = \mathrm{i}\frac{m^2}{16\pi^2}\left[\frac{1}{\epsilon} + 1 - \gamma_E - \log\left(\frac{m^2}{4\pi}\right) + O(\epsilon)\right]$$

Virtual corrections



$$\mathcal{M}^{(1)}_{q\bar{q}} = \frac{e^2 q_q}{s} \Big(\overline{v}(p_2)\gamma_\mu u(p_1) \Big) \Big(\overline{u}(p_3)\Gamma_\mu v(p_4) \Big)$$

$$\overline{u}(p_3)\Gamma_\mu v(p_4) = C_F g_s^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - \lambda^2} \overline{u}(p_3)\gamma^\alpha \frac{\not{k} + \not{p}_3}{(p_3 + k)^2} \gamma^\nu \frac{\not{k} - \not{p}_4}{(p_4 - k)^2} \gamma_\alpha v(p_4)$$

- Integration with multiple propagators.
- Algebra of gamma matrices in d dimensions.

Feynman parameters

- Feynman parameters

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^2 dy \cdots \int_0^1 dz \frac{\delta(1-x-y-z)}{(xA+yB+zC)^3}$$

- We can combine the denominators as follows

$$\int \frac{d^dk}{(2\pi)^d} \frac{N(k)}{(k^2-\lambda^2)(p_3+k)^2(p_4-k)^2}$$
$$= \int_0^1 dx\, dy\, dz \int \frac{d^dk}{(2\pi)^d} \frac{N(k)}{[\underbrace{\underbrace{(x+y+z)}_{=1}k^2 + 2yp_3\cdot k - 2zp_4\cdot k}_{\text{complete the square}} - x\lambda^2]^3}$$

Feynman parameters

$$\int \frac{d^d k}{(2\pi)^d} \frac{N(k)}{(k^2-\lambda^2)(p_3+k)^2(p_4-k)^2}$$

$$= \int_0^1 dx\, dy\, dz \int \frac{d^d k}{(2\pi)^d} \frac{N(k)}{[(\underbrace{k+yp_3-zp_4}_{\equiv k'})^2 + \underbrace{2yzp_3\cdot p_4}_{=yzs} - x\lambda^2]^3}$$

$$= \int_0^1 dx\, dy\, dz \int \frac{d^d k'}{(2\pi)^d} \frac{N(k'-yp_3+zp_4)}{[k'^2 - \underbrace{(x\lambda^2 - yzs)}_{\Delta}]^3}$$

- We could almost perform the integration w.r.t. k'

Dealing with numerators

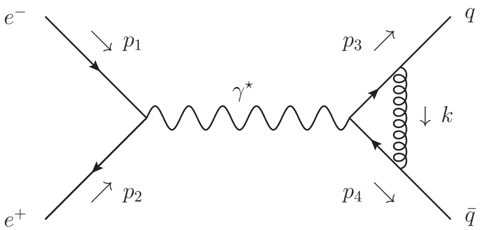
- Tensor integrals

$$\int \frac{d^d k'}{(2\pi)^d} \frac{k'^{\mu}}{(k'^2 - \Delta)^n} = 0 \qquad \int \frac{d^d k'}{(2\pi)^d} \frac{k'^{\mu} k'^{\nu}}{(k'^2 - \Delta)^n} = \frac{g^{\mu\nu}}{d} \int \frac{d^d k'}{(2\pi)^d} \frac{k'^2}{(k'^2 - \Delta)^n}$$

- Gamma matrices in d dimensions

$$\gamma^\mu \gamma_\mu = g_{\mu\nu} \gamma^\mu \gamma^\nu = \frac{g_{\mu\nu}}{2} \{\gamma^\mu \, \gamma^\nu\} = \frac{g_{\mu\nu}}{2} (2 g^{\mu\nu} \mathbb{1}) = d \mathbb{1}$$
$$\gamma^\mu \gamma^\alpha \gamma_\mu = \gamma^\mu (2 g^\alpha{}_\mu - \gamma_\mu \gamma^\alpha) = (2 - d) \gamma^\alpha$$
$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 2 \gamma^\beta \gamma^\alpha - (d - 2) \gamma^\alpha \gamma^\beta$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = 2 \gamma^\rho \gamma^\sigma \gamma^\nu - 2 \gamma^\nu \gamma^\sigma \gamma^\rho - (d - 2) \gamma^\nu \gamma^\rho \gamma^\sigma$$

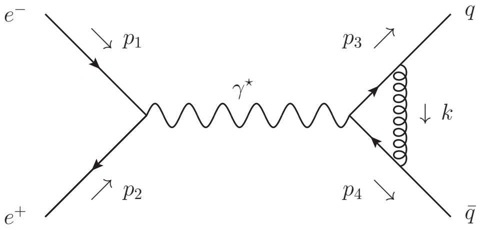
$e^+e^- \to q\bar{q}$: virtual corrections

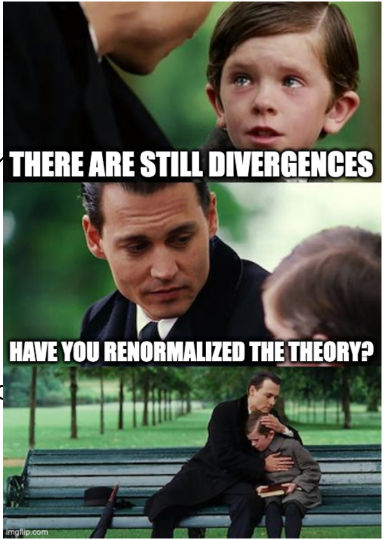


$$\mathcal{M}^{(1)}_{q\bar{q}} = f(s)\mathcal{M}^{(0)}_{q\bar{q}}$$

$$f(s) = \frac{C_F \alpha_s}{4\pi}\left[\frac{1}{\epsilon} - 4 - \frac{2\pi^2}{3} - 3\log\left(\frac{\lambda^2}{-s - i\delta}\right) - \log^2\left(\frac{\lambda^2}{-s - i\delta}\right) + O(\epsilon)\right]$$

$e^+e^- \to q\bar{q}$: virtual corrections





$$f(s) = \frac{C_F \alpha_s}{4\pi} \left[\frac{1}{\epsilon} - 4 - \frac{2\pi^2}{3} - 3 \log\left(\frac{\lambda^2}{-s - i\delta} \right) - \mathrm{lo}\right.$$

Renormalization

- The quantities initially appearing in the Lagrangian, representing such things as the electron's electric charge and mass, as well as the normalizations of the quantum fields themselves, **do not actually correspond to the physical constants** that are measure.

$$\mathcal{L}_{\mathrm{QED}} = \overline{\psi}(\mathrm{i}\not{\partial} - m)\psi - e\overline{\psi}\not{A}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- IDEA: Physical constants defined by a measurement!

Renormalization

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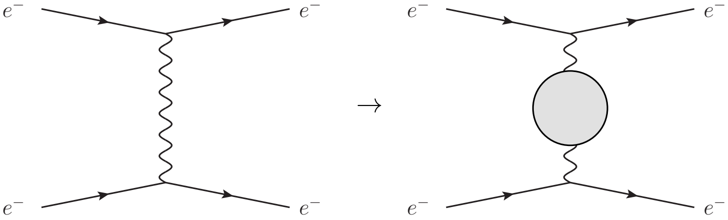
$$\mathcal{L}_{\text{QED}} = \overline{\psi}(\mathrm{i}\not{\partial} - m_B)\psi - e_B\overline{\psi}\not{A}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- IDEA: Physical constants defined by a measurement!
- What we write in the Lagrangian are just symbols! **Bare quantities!**
  - bare electric charge e_B
  - bare mass m_B.
  - bare fileds A_B^μ and ψ_B.

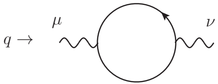
Charge renormalization

- The scattering of two electrons in the limit $r \to \infty$ should reduce to the usual Coulomb interactions in the non-relativistic limit



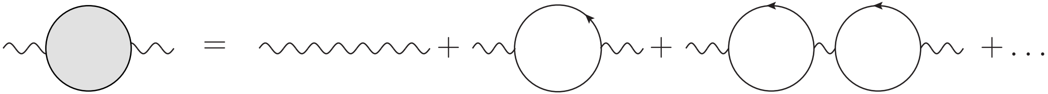
$$\tilde{V}(q^2) \simeq -\frac{e^2}{q^2}$$

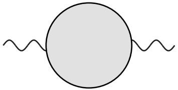
- Vacuum polarization



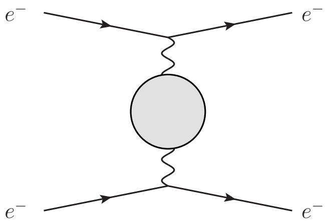
$$= i\Pi(q^2)(g^{\mu\nu}q^2 - q^\mu q^\nu)$$

- The photon propagator summed to all orders





$$= \frac{-ig^{\mu\nu}}{q^2} + \frac{-ig^{\mu\rho}}{q^2}\Big(\Pi(q^2)(g_{\rho\sigma}q^2 - q_\rho q_\sigma)\Big)\frac{-ig^{\sigma\nu}}{q^2} \simeq \frac{-ig^{\mu\nu}}{q^2[1-\Pi(q^2)]} + \dots$$



$$\tilde{V}(q^2) \simeq -\frac{e_B^2}{q^2[1-\Pi(q^2)]}$$

- The physical or **renormalized electric charge e** is determined by the limit $q^2 \to 0$ (i.e. $r \to \infty$)

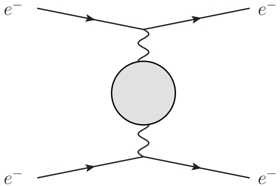
$$\tilde{V}(q^2) \simeq -\frac{e_B^2}{q^2[1-\Pi(q^2)]} \stackrel{!}{=} -\frac{e^2}{q^2}$$

- Renromalization of the electric charge

$$e_B^2 = \underbrace{e^2}_{\text{ren. charge}} \times \underbrace{[1-\Pi(0)]}_{\text{ren. constant}} \Leftrightarrow \alpha_B = Z_\alpha\, \alpha \text{ with } \alpha = \frac{e^2}{4\pi}$$

Charge renormalization

- The bare charge is cannot be observed.
- The scattering of two particle at a different scale $q^2 \neq 0$ involve



$$\mathcal{M} \simeq \frac{e_B^2}{q^2[1-\Pi(q^2)]}\left(\bar{u}\gamma^\mu u\right)\left(\bar{v}\gamma^\mu v\right)$$

$$\simeq \frac{e^2}{q^2}\left[1+\Pi(q^2)-\Pi(0)\right]\left(\bar{u}\gamma^\mu u\right)\left(\bar{v}\gamma^\mu v\right)+O(e^2)$$

- The renormalization yields the cancellation of UV divergences

$$\Pi(q^2) = -\frac{e^2}{12\pi^2}\left[\frac{1}{\epsilon}+\text{finite terms}\right]$$

Renormalization constants

- The fields, masses and coupling constants in the Lagrangian are **bare** quantities.

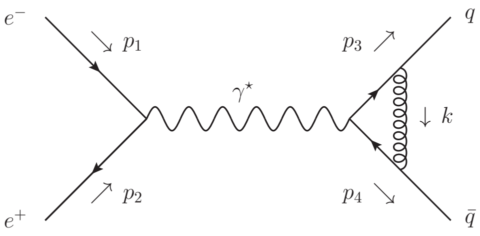
Renormalization

$$e_B = Z_e\, e \qquad m_B = Z_m\, m$$

$$\Psi_B = Z_2 \psi \qquad A_B^\mu = Z_3 A^\mu$$

- Renormalization constants Z_i contain $1/\epsilon$ poles which compensate the poles from virtual corrections.
- They are computed order by order in perturbaiton theory

$$Z_i = 1 + \delta Z_i = + \frac{\alpha}{\pi} \delta Z_i^{(1)} + + \left(\frac{\alpha}{\pi}\right)^2 \delta Z_i^{(2)} + \dots$$



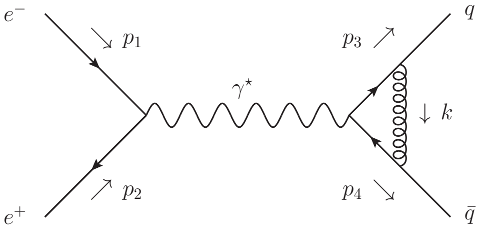
- Wave-function renormalization

$$\mathcal{M}_{q\bar{q}} = (\sqrt{Z_2})^2\left[\mathcal{M}^{(0)}_{q\bar{q}} + \mathcal{M}^{(1)}_{q\bar{q}} + \ldots\right]$$
$$= \mathcal{M}^{(0)}_{q\bar{q}} + \underbrace{\mathcal{M}^{(1)}_{q\bar{q}} + \delta Z_2 \mathcal{M}^{(0)}_{q\bar{q}}}_{O(\alpha_s)}$$
$$= \mathcal{M}^{(0)}_{q\bar{q}}\left[1 + f(s) + \delta Z_2\right]$$

$$f(s) = \frac{C_F\alpha_s}{4\pi}\left[+\frac{1}{\epsilon} - 4 - \frac{2\pi^2}{3} - 3\log\left(\frac{\lambda^2}{-s-\mathrm{i}\delta}\right) - \log^2\left(\frac{\lambda^2}{-s-\mathrm{i}\delta}\right) + O(\epsilon)\right]$$

$$\delta Z_2 = \frac{C_F\alpha_s}{4\pi}\left[-\frac{1}{\epsilon} + \frac{1}{2}\right] + O(\alpha_s^2)$$

Total cross section: virtual contribution



$$\frac{\alpha_s}{\pi}\sigma_{q\bar{q}}^{(1)} = \frac{(2\pi^4)}{2s}\int 2\mathrm{Re}\left(\mathcal{M}_{q\bar{q}}^{(1)}\mathcal{M}_{q\bar{q}}^{(0)\,\dagger}\right)d\Phi_2$$
$$= \frac{(2\pi^4)}{2s}\int 2\mathrm{Re}\left(f(s)+\delta Z_2^{(1)}\right)\overline{|\mathrm{M}_{q\bar{q}}^{(0)}|^2}d\Phi_2$$
$$= 2\mathrm{Re}\left(f(s)+\delta Z_2^{(1)}\right)\sigma_{q\bar{q}}^{(0)}$$

Virtual correction for $e^+e^- \to q\bar{q}$

$$\frac{\alpha_s}{\pi}\sigma_{q\bar{q}}^{(1)} = \frac{C_F\alpha_s}{4\pi}\left[-2\log^2\left(\frac{\lambda^2}{s}\right) - 6\log\left(\frac{\lambda^2}{s}\right) - 7 + \frac{2\pi^2}{3}\right]\sigma_{q\bar{q}}^{(0)}$$

Exercise III

GOAL: Compute the total cross section for real emission at various values of λ

$$\sigma^{(0)}_{q\bar{q}g}(\lambda) = \frac{(2\pi)^4}{2s} \int_{x_1^{\min}}^{x_1^{\max}} dx_1 \cdots \int_{x_k^{\min}}^{x_k^{\max}} dx_k \, \overline{|\mathcal{M}|^2} \left| \frac{d\Phi_n}{d\vec{x}} \right|$$

- `M2 = SquaredMatrixElement(k, p3, p4, s, lambda)`
- `[p3, p4, k] = PhaseSpacePoint(x1,..., xk, s, lambda)`
- `dPhi3 = DifferentialPhaseSpaceVolume(x1,...,xk, s, lam2)`

Exercise III

Task I Evaluate the contribution of real emission to the total cross section.

Use different values of λ:

$$\lambda^2 = 10^{-1}, \ldots, 10^{-6}$$

Task II Add to the real emission the contribution of virtual correction $\sigma^{(1)}_{q\bar{q}}$.
Check that the sum of the two is independent on λ for $\lambda \to 0$, i.e. the cancellation of IR divergences

$$\sigma = \sigma^{(0)}_{q\bar{q}} \left(1 + \frac{3C_F\alpha_s}{4\pi}\right) = \frac{4\pi\alpha_{\mathrm{em}}}{3s} N_c q_q^2 \left(1 + \frac{3C_F\alpha_s}{4\pi}\right)$$

Task III Generate 10000 events for real emission with $\lambda^2 = 10^{-3}$. Create histogram for the angular separation between gluon and quarks and the gluon energy.