Loops in inclusive determinations

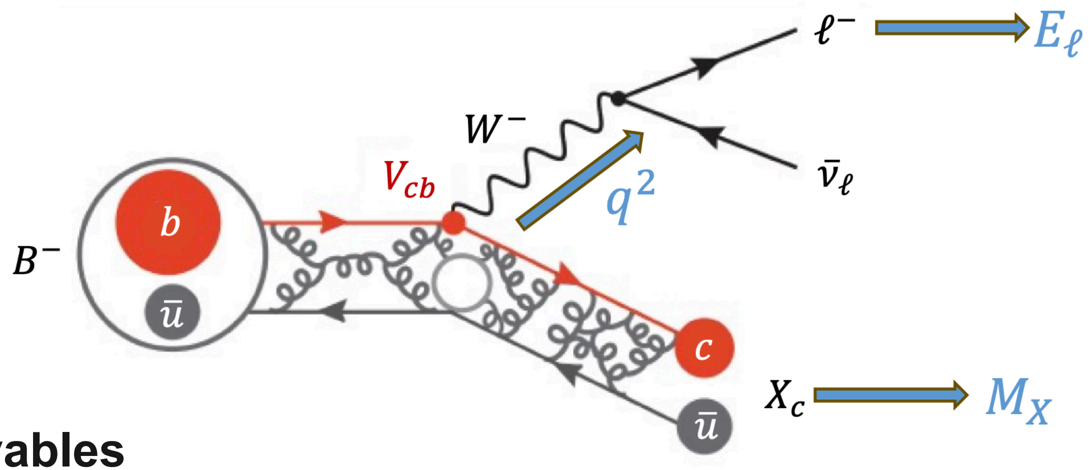
Matteo Fael (CERN)

2023 Belle II Physics Week - KEK - 31 Ott. 2023

MARIE CURIE ACTIONS

Funded by
the European Union

Extraction of V_{cb} from inclusive $B \to X_c l \nu$ decays



Observables

- Total rate $\Gamma_{\mathrm{sl}} = \Gamma(B \to X_c \ell \bar{\nu}_\ell)$
- Moments of the differential distribution of an observables O

$$\langle (O)^n \rangle_{\mathrm{cut}} = \int_{\mathrm{cut}} (O)^n \frac{\mathrm{d}\Gamma}{\mathrm{d}O} \mathrm{d}O \Big/ \int_{\mathrm{cut}} \frac{\mathrm{d}\Gamma}{\mathrm{d}O} \mathrm{d}O$$

- $O = E_\ell$: energy of the charged lepton in the B rest frame
- $O = M_X^2$: hadronic invariant mass
- $O = q^2$: leptonic invariant mass

see talks by M. Bordone, K. Vos & P. Gambino

Heavy Quark Expansion

Double series expansion in the strong coupling constant α_s and power suppressed terms $\Lambda_{\mathrm{QCD}}/m_b$

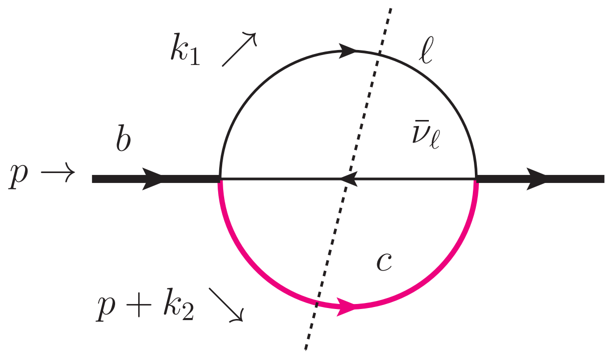
- $\rho = m_c/m_b$
- Total rate

$$\Gamma_{\mathrm{sl}} = \frac{G_F^2 m_b^5 A_{\mathrm{ew}}}{192\pi^3} |V_{cb}|^2 \left[\left(1 - \frac{\mu_\pi^2}{2m_b^2}\right)\left(X_0(\rho) + \frac{\alpha_s}{\pi} X_1(\rho) + \left(\frac{\alpha_s}{\pi}\right)^2 X_2(\rho) + \left(\frac{\alpha_s}{\pi}\right)^3 X_3(\rho) + \ldots\right) \right.$$
$$\left. + \left(\frac{\mu_G^2}{m_b^2} - \frac{\rho_D^3}{m_b^3}\right)\left(g_0(\rho) + \frac{\alpha_s}{\pi} g_1(\rho) + \ldots\right) + \frac{\rho_D^3}{m_b^3}\left(d_0(\rho) + \frac{\alpha_s}{\pi} d_1(\rho) + \ldots\right) + O\left(\frac{1}{m_b^4}\right)\right]$$

- Moments of differential distribution for some observable O

$$\langle O^n \rangle_{\mathrm{cut}} = (m_b)^{mn} \left[X_0^{(O,n)} + \frac{\alpha_s}{\pi} X_1^{(O,n)} + \left(\frac{\alpha_s}{\pi}\right)^2 X_2^{(O,n)} + \frac{\mu_\pi^2}{m_b^2}\left(p_0^{(O,n)} + \frac{\alpha_s}{\pi} p_1^{(O,n)} + \ldots\right) \right.$$
$$\left. + \frac{\mu_G^2}{m_b^2}\left(g_0^{(O,n)} + \frac{\alpha_s}{\pi} g_1^{(O,n)} + \ldots\right) + \frac{\rho_D^3}{m_b^3}\left(d_0^{(O,n)} + \frac{\alpha_s}{\pi} d_1^{(O,n)} + \ldots\right) + \frac{\rho_{LS}}{m_b^2}\left(l_0^{(O,n)} + \frac{\alpha_s}{\pi} l_1^{(O,n)} + \ldots\right) + O\left(\frac{1}{m_b^4}\right)\right]$$

Why higher order corrections?



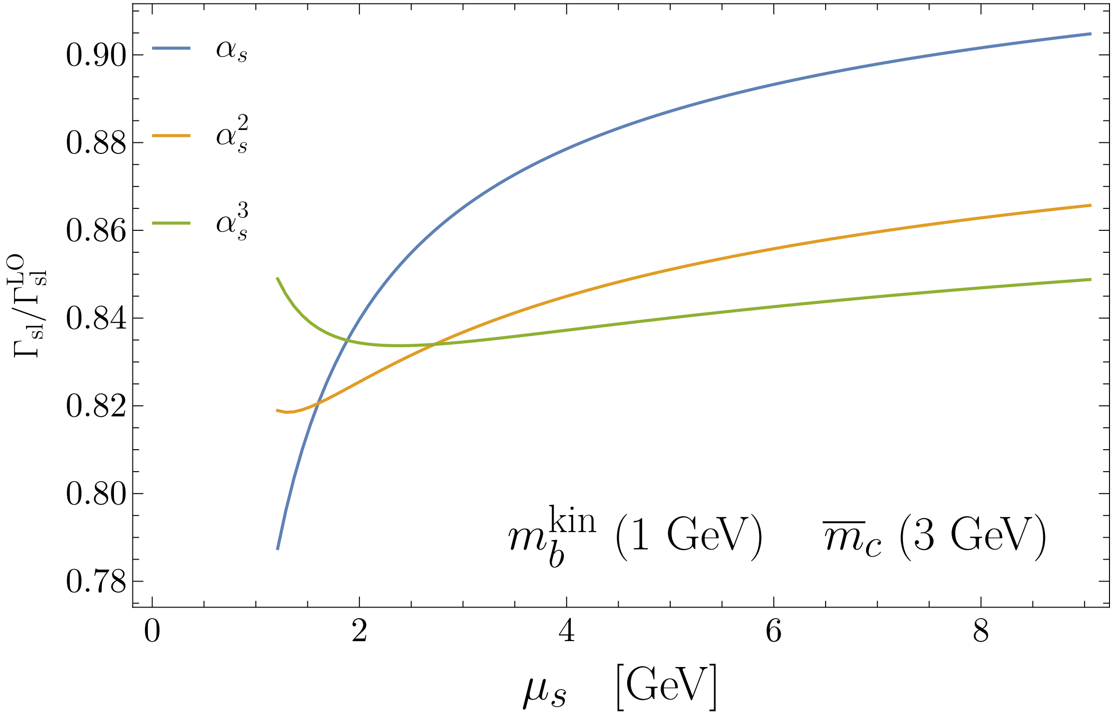
- Missing higher-order terms limit the prediction

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 A_{\rm ew}}{192\pi^3} |V_{cb}|^2 \times \left(X_0(\rho) + \frac{\alpha_s}{\pi} X_1(\rho) + \left(\frac{\alpha_s}{\pi}\right)^2 X_2(\rho) + \left(\frac{\alpha_s}{\pi}\right)^3 X_3(\rho) + \dots \right)$$

- $\alpha_s(\mu_s), m_b^{\rm kin}(\mu_{WC}), \overline{m}_c(\mu_c), \mu_G^2(\mu_g)$
- Assess theoretical uncertainties

$$m_b/2 \le \mu_s \le 2m_b$$

- Reduce impact of theory correlations



MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

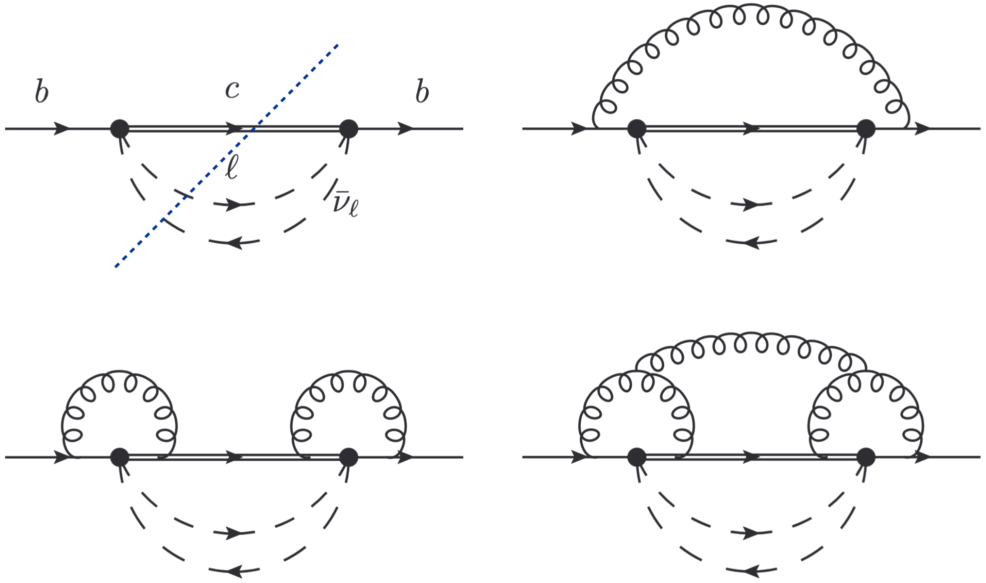
Order in α_s

3

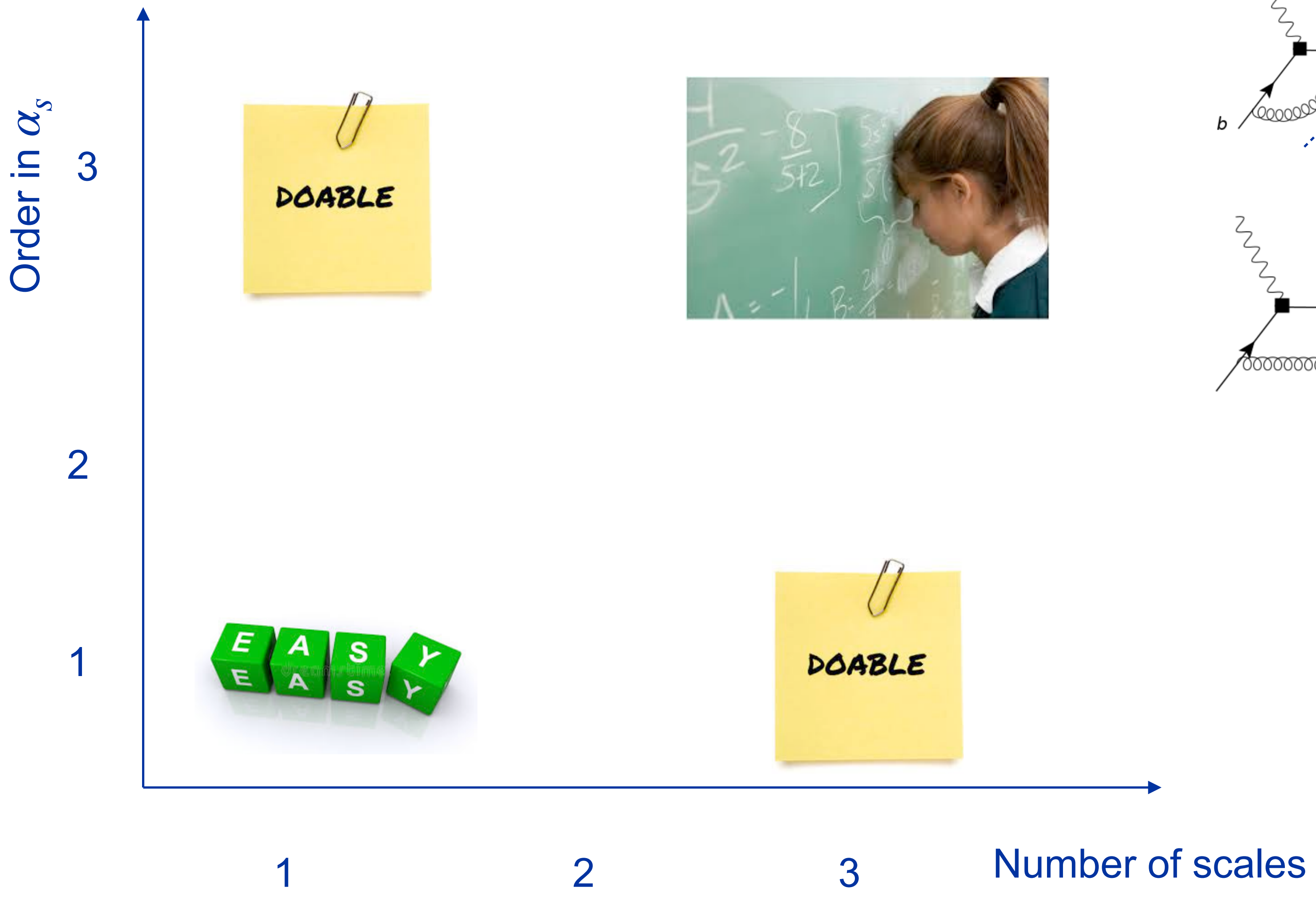
2

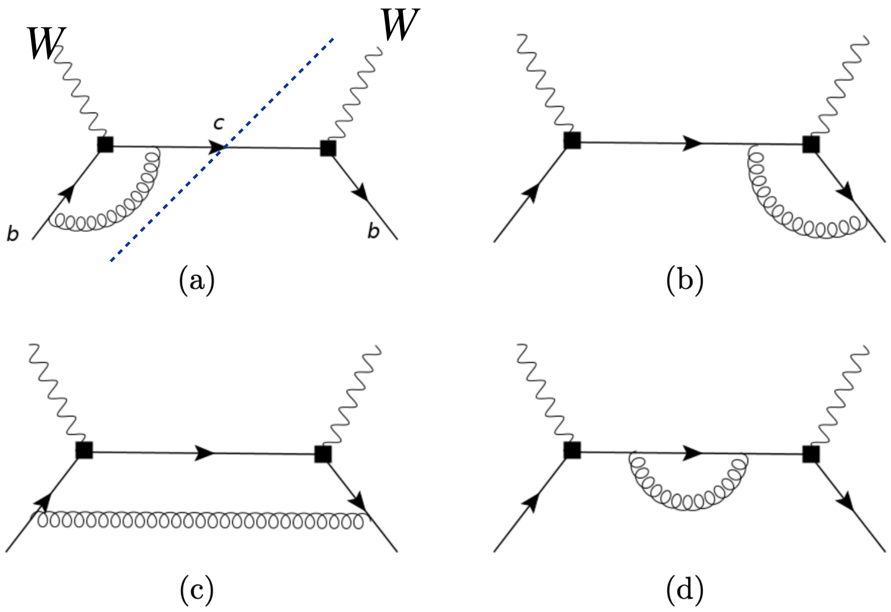
1

1 | 2 | 3 | Number of scales



- **Scales:**
  - $m_b \to 1$
  - m_c





- **Scales:**
  - $m_b \to 1$
  - m_c
  - q^2
  - $q_0 = p_b \cdot q$

State-of-the art

- **Total Rate**
  - NNLO Czarnecki, Pak, *Phys.Rev.D* 78 (2008) 114015, *Phys.Rev.Lett.* 100 (2008) 241807
  - N3LO $(b \to c)$ **MF**, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003.
  - N3LO $(b \to u)$ **NEW** **MF**, Usovitsch, hep-ph/2310.03685
- **M_X and E_l moments**
  - NLO differential rate Aquila, Gambino, Ridolfi, Uraltsev, Nucl.Phys.B 719 (2005) 77
  - NNLO for moments with $E_{\rm cut} < E_l$, numerical results for specific $E_{\rm cut}$ and $\rho = m_c/m_b$ Biswas, Melnikov, JHEP 02 (2010) 089; Gambino, JHEP 09 (2011) 055. Gambino, JHEP 09 (2011) 055.
  - NLO for μ_π^2 and μ_G^2 Alberti, Gambino, Nandi, *Nucl.Phys.B* 870 (2013) 16, *JHEP* 01 (2014) 147
  - N3LO for moments without cuts **MF**, Schönwald, Steinhauser, JHEP 08 (2022) 039.
- **q^2 moments with a lower cut on q^2**
  - NLO up to ρ_D^3 Moreno, Mannel, Pivovarov, *Phys.Rev.D 105 (2022) 5, 054033*
  - NNLO for moments with $q^2_{\rm cut} \leq q^2$ **NEW** **MF**, Herren, in preparation
- **QED effects** Bordone, Gambino, Haisch, Piccione, hep-ph-2309.02849

NNLO corrections to q² spectrum

MF, Herren, in preparation

GOAL:
$$\frac{d\Gamma}{d\hat{q}^2} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[F_0(\rho, \hat{q}^2) + \frac{\alpha_s}{\pi} F_1(\rho, \hat{q}^2) + \left(\frac{\alpha_s}{\pi}\right)^2 F_2(\rho, \hat{q}^2) \right] + O\left(\frac{1}{m_b^2}\right)$$

$$d\Gamma \simeq \delta(p_L^2 - q^2) W_{\mu\nu} L^{\mu\nu} \, d\Phi_2(p_b; p_L, p_X) \, d\Phi_2(p_L; p_\ell, p_\nu) \, dq^2$$

with $p_L = p_l + p_\nu$

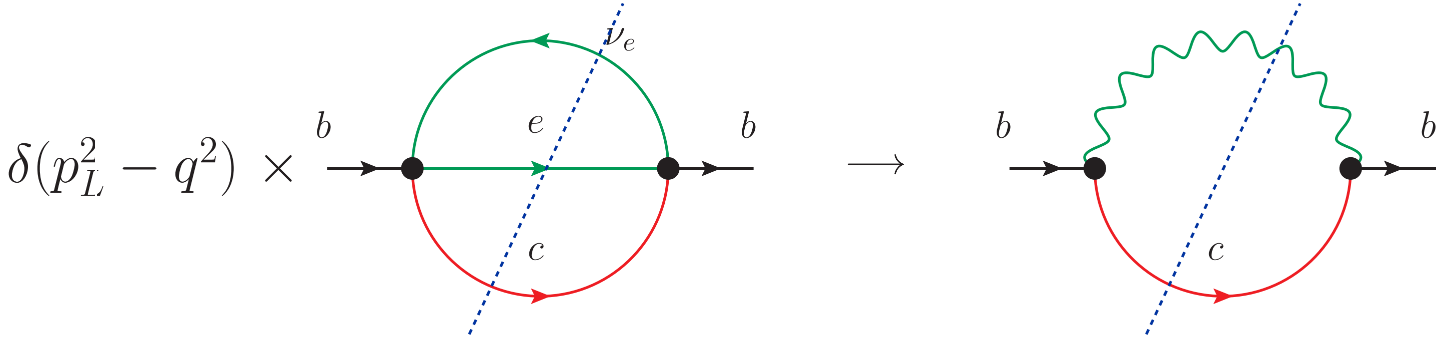
Method employed at NLO in Moreno, Mannel, Pivovarov, *Phys.Rev.D 105 (2022) 5, 054033*

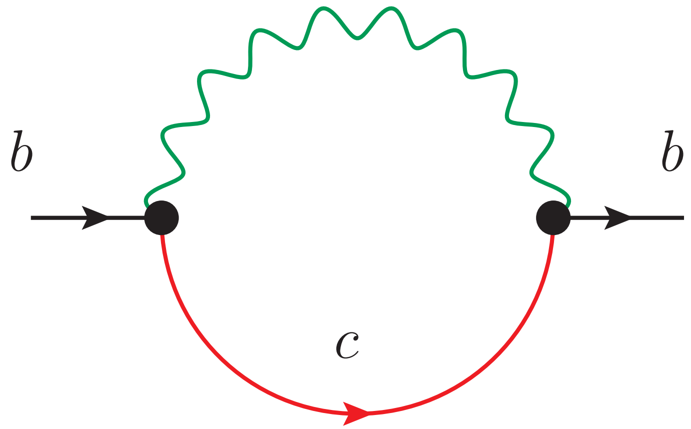
Integration of electron and neutrino phase-space (with $p_L = p_l + p_\nu$)

$$\mathscr{L}^{\mu\nu}(p_L) = \int L^{\mu\nu} d\Phi_2(p_L; p_l, p_\nu) = \frac{1}{384\pi^5}\left(1 - \frac{m_\ell^2}{p_L^2}\right)^2 \left[\left(1 + \frac{2m_\ell^2}{p_L^2}\right) p_L^\mu p_L^\nu - g^{\mu\nu} p_L^2 \left(1 + \frac{m_\ell^2}{2p_L^2}\right)\right]$$

Inverse unitarity $$\delta(p_L^2 - q^2) \to \frac{1}{2\pi i}\left[\frac{1}{p_L^2 - q^2 - i0} - \frac{1}{p_L^2 - q^2 + i0}\right]$$

Anastasiou, Melnikov, Nucl.Phys.B 646 (2002) 220

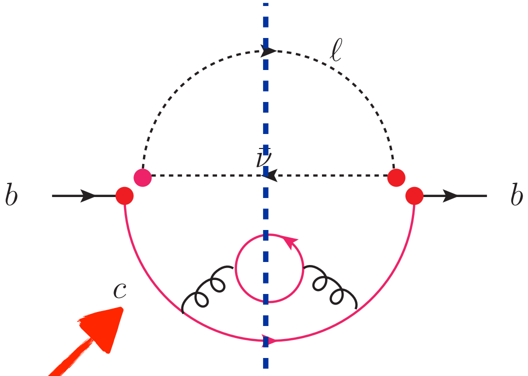




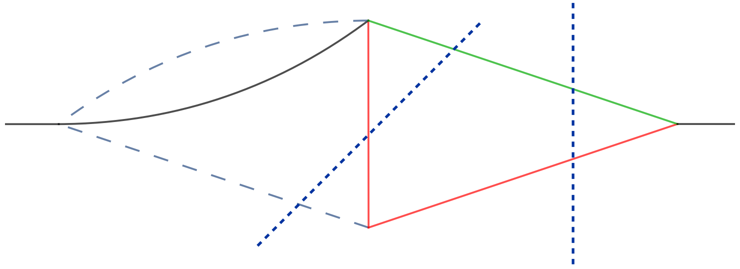
$$\rightarrow \sum_n \underbrace{\frac{p_n(\hat{q}^2, \rho; \epsilon)}{q_n(\hat{q}^2, \rho; \epsilon)}}_{\text{polynomials}} \underbrace{J_n(\hat{q}^2, \rho; \epsilon)}_{\text{master integrals}} \qquad \epsilon = \frac{4-d}{2}$$

- **98 master integrals**
- **Neglect cuts through 3 charm**

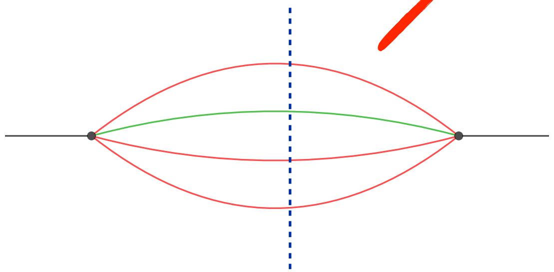
see also: Egner, **MF**, Schönwald, Steinhauser, HEP 09 (2023) 112



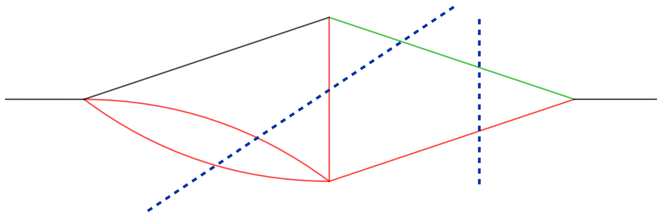
Three charm cuts



One charm cuts



Three charm cuts



One charm cuts

- **Differential equations**
  - Canonical form

    Henn, Phys.Rev.Lett. 110 (2013) 251601
    Lee, Comput.Phys.Commun. 267 (2021) 108058

$$\frac{\partial \vec{J}}{\partial \rho} = \epsilon \hat{M}_\rho(\hat{q}^2, \rho) \vec{J}(\hat{q}^2, \rho, \epsilon)$$

$$\frac{\partial \vec{J}}{\partial \hat{q}^2} = \epsilon \hat{M}_{q^2}(\hat{q}^2, \rho) \vec{J}(\hat{q}^2, \rho, \epsilon)$$

  - Boundary conditions evaluated with AMFlow and subsequent PSLq

    Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565

  - Analytic solution in terms of Generalised Polylogarithms (GPLs)

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

Examples:

$$G(0; z) = \log(z) \qquad G(x, z) = \log\left(1 - \frac{z}{x}\right)$$

$$G(\underbrace{0,\ldots,0}_{n}; z) = \frac{\log^n(z)}{n!} \qquad G(\underbrace{0,\ldots,0}_{n}, x, z) = -\,\mathrm{Li}_n\left(\frac{z}{x}\right)$$

Fast numerical evaluation with GiNaC and PolyLogTools

http://www.ginac.de
Duhr, Dulat, JHEP 08 (2019) 135

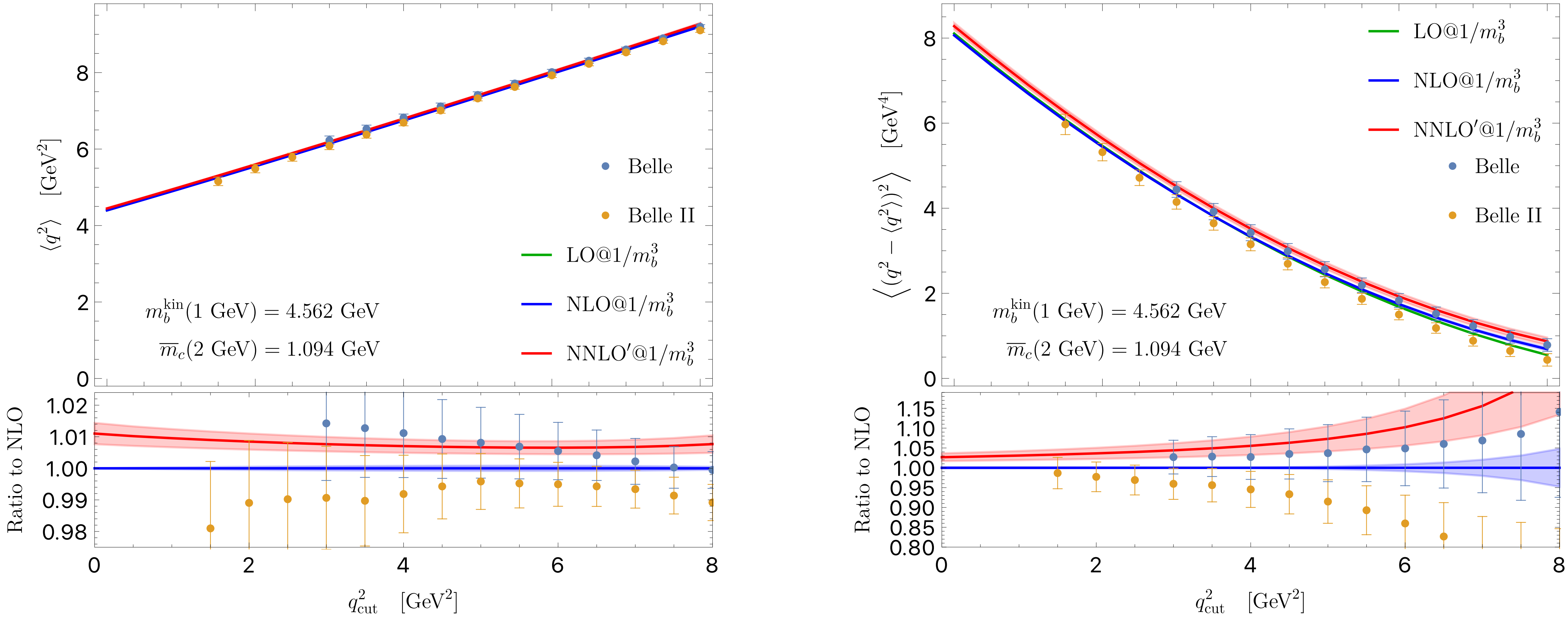
$$G\left(x, \frac{1+x^2}{x}, x, \frac{1}{x}; z\right)\Bigg|_{x=1/2,\, z=1/3} = 0.00151860208899279\ldots$$

$$\frac{d\Gamma}{d\hat{q}^2} = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2 \left[F_0(\rho, \hat{q}^2) + \frac{\alpha_s}{\pi} F_1(\rho, \hat{q}^2) + \left(\frac{\alpha_s}{\pi}\right)^2 F_2(\rho, \hat{q}^2) \right] + O\left(\frac{1}{m_b^2}\right)$$

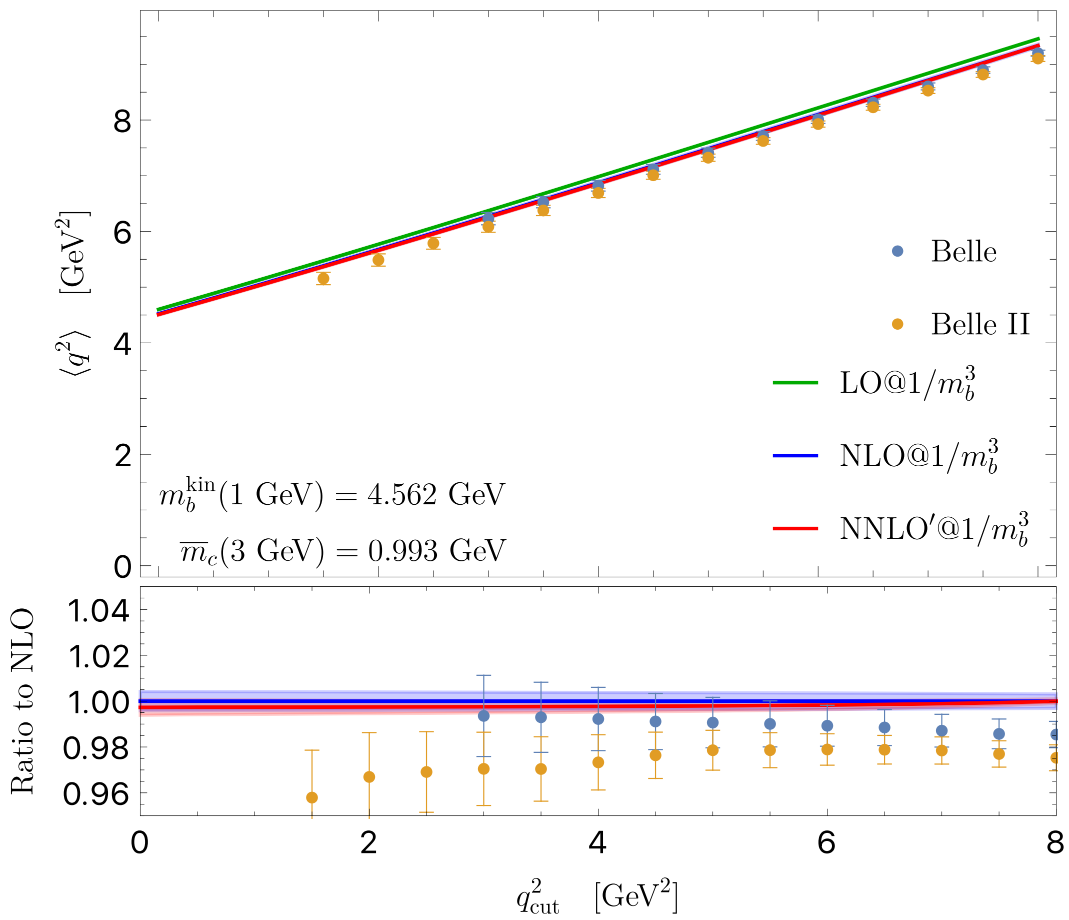
Normalised moments
$$\langle (q^2)^n \rangle_{q^2_{\mathrm{cut}}} = \int_{q^2 > q^2_{\mathrm{cut}}} (q^2)^n \frac{d\Gamma}{dq^2} dq^2 \Bigg/ \int_{q^2 > q^2_{\mathrm{cut}}} \frac{d\Gamma}{dq^2} dq^2$$

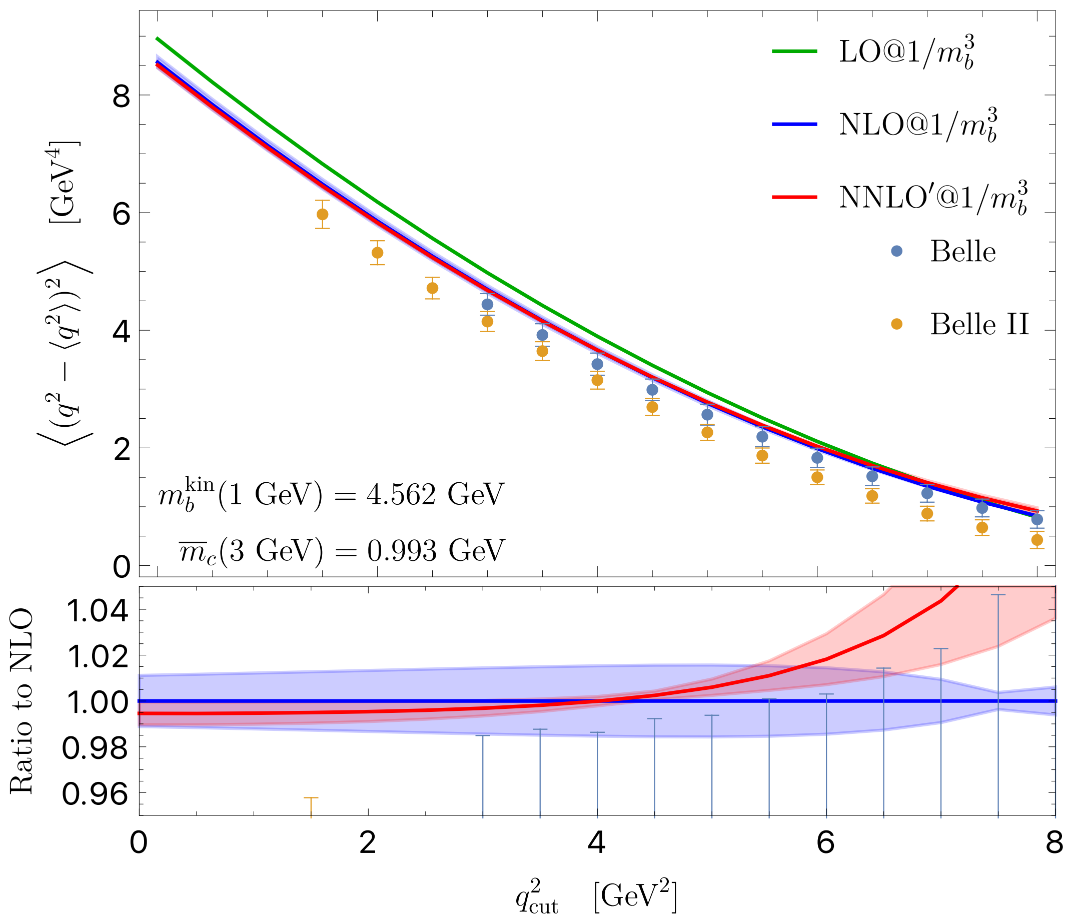
Centralized moments
$$\left\langle (q^2 - \langle q^2 \rangle)^n \right\rangle_{q^2_{\mathrm{cut}}} = \sum_{i=1}^{n} \binom{n}{i} \langle q^{2i} \rangle \left(\langle q^2 \rangle \right)^{n-i}$$

Change of mass scheme
$$m_b^{\mathrm{OS}}, m_c^{\mathrm{OS}} \to m_b^{\mathrm{kin}}, \overline{m}_c$$



HQE parameters from: Bernlochner et al, JHEP 10 (2022) 068

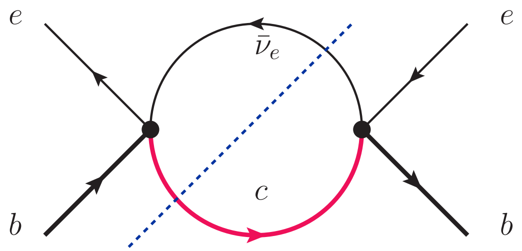




HQE parameters from: Bernlochner et al, JHEP 10 (2022) 068

Revisiting NNLO corrections to E$_l$ moments

- Similar strategy applies to energy spectrum of the electron:



$$\simeq \frac{1}{E_l}\frac{d\Gamma}{dE_l} \quad \rightarrow \quad \langle E_l^n \rangle_{E_{\rm cut}} = \int_{E_l > E_{\rm cut}} (E_l)^n \frac{d\Gamma}{dE_l} dE_l$$

- The master integrals depends on two scales: $\rho = m_c/m_b$ and $E_l = p_b \cdot p_l/m_b$.
- At NLO there are 9 master integrals.
- Perfect numerical agreement with integration of differential rate.

Aquila, Gambino, Ridolfi, Uraltsev, Nucl.Phys.B 719 (2005) 77

- Possibility to extend the calculation at NNLO under study.

MF, Herren, Schönwald, work in progress

Why only E_l moments?

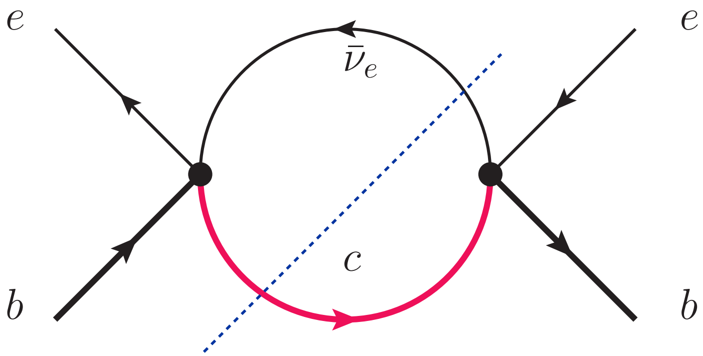
$$\langle (M_X^2)^n \rangle_{E_\mathrm{cut}} = \int_{E_l > E_\mathrm{cut}} M_X^{2n} \frac{d\Gamma}{dq^2\, dq_0\, dE_l} dq^2\, dq_0\, dE_l \Big/ \int_{E_l > E_\mathrm{cut}} \frac{d\Gamma}{dE_l} dE_l$$

$$\langle (q^2)^n \rangle_{E_\mathrm{cut}} = \int_{E_l > E_\mathrm{cut}} q^{2n} \frac{d\Gamma}{dq^2\, dq_0\, dE_l} dq^2\, dq_0\, dE_l \Big/ \int_{E_l > E_\mathrm{cut}} \frac{d\Gamma}{dE_l} dE_l$$

Integration order does not matter ….

$$dM_X^{2n} = (M_X^2)^n \times W_{\mu\nu} L^{\mu\nu} d\Phi_2(p_b; p_l, p_\nu, p_X)$$

$$= (M_B^2 + q^2 - 2M_B\, q_0)^n \times$$



Why only E_l moments?

$$\langle (M_X^2)^n \rangle_{E_{\rm cut}} = \int_{E_l > E_{\rm cut}} \frac{dM_X^{2n}}{dE_l}\, dq^2\, dq_0\, dE_l \Bigg/ \int_{E_l > E_{\rm cut}} \frac{d\Gamma}{dE_l}\, dE_l$$

$$\langle (q^2)^n \rangle_{E_{\rm cut}} = \int_{E_l > E_{\rm cut}} \frac{dq^{2n}}{dE_l}\, dq^2\, dq_0\, dE_l \Bigg/ \int_{E_l > E_{\rm cut}} \frac{d\Gamma}{dE_l}\, dE_l$$

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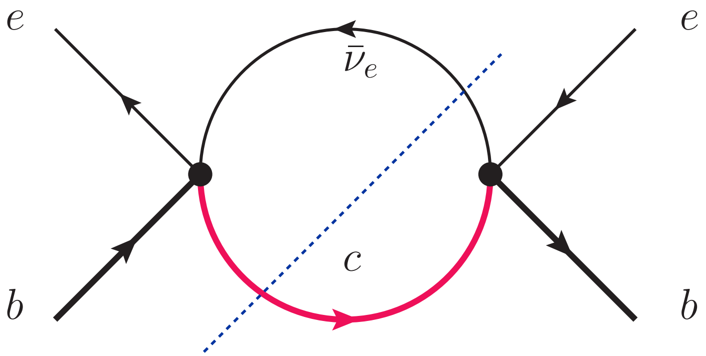
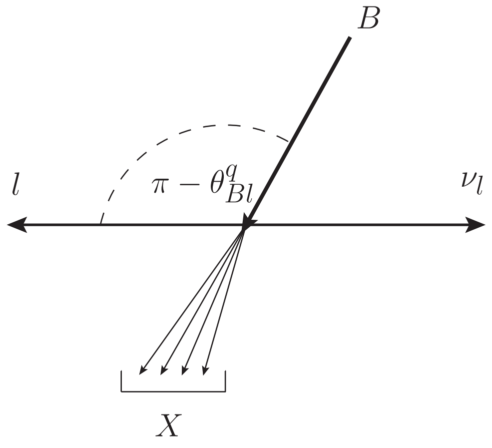
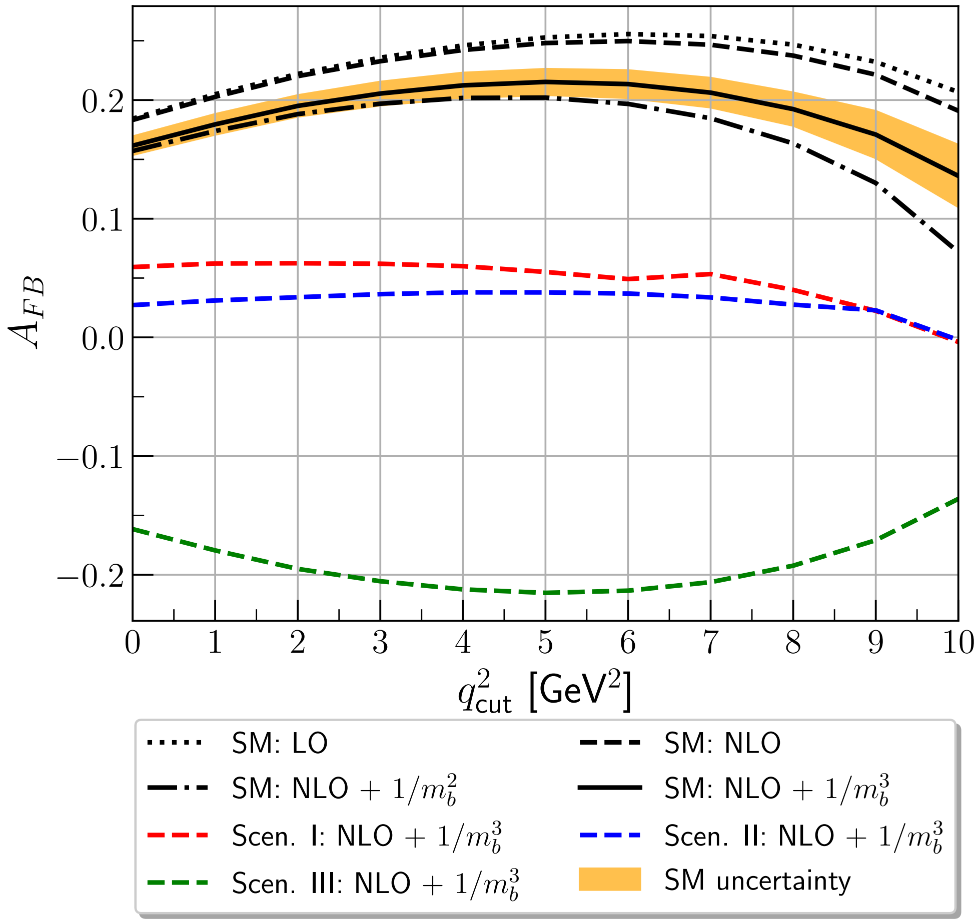


Forward-backward asymmetry

$$A_{FB} = \frac{\int_{-1}^{0} \frac{d\Gamma}{dz} - \int_{-1}^{0} \frac{d\Gamma}{dz}}{\int_{-1}^{0} \frac{d\Gamma}{dz} + \int_{-1}^{0} \frac{d\Gamma}{dz}} \qquad z = \cos\theta = \frac{v \cdot p_\nu - v \cdot p_\ell}{\sqrt{(v\cdot q)^2 - q^2}}$$

$$A_{FB}(q^2_{\text{cut}} = 4\,\text{GeV}) = 24.6\,(1 - 0.019|_{\alpha_s} - 0.12|_{\text{pow}}) \times 10^{-2}$$





MF, Rahimi, Vos, *JHEP* 02 (2023) 086

State-of-the art

- **Total Rate**
  - NNLO Czarnecki, Pak, *Phys.Rev.D* 78 (2008) 114015, *Phys.Rev.Lett.* 100 (2008) 241807
  - N3LO ($b \to c$) **MF**, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003.
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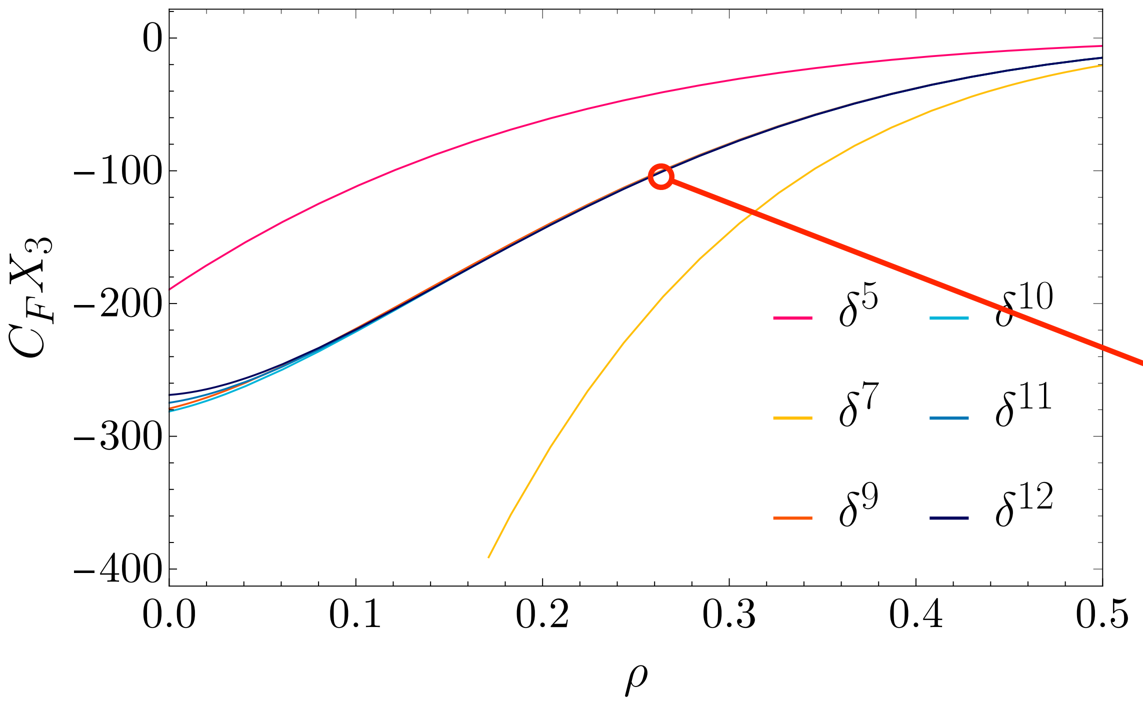
Total rate of $b \to u$



$$\delta = 1 - \frac{m_c}{m_b}$$

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 A_{\rm ew}}{192\pi^3} |V_{cb}|^2 \left(X_0(\rho) + C_F \sum_n \left(\frac{\alpha_s}{\pi}\right)^n X_n(\rho) \right)$$

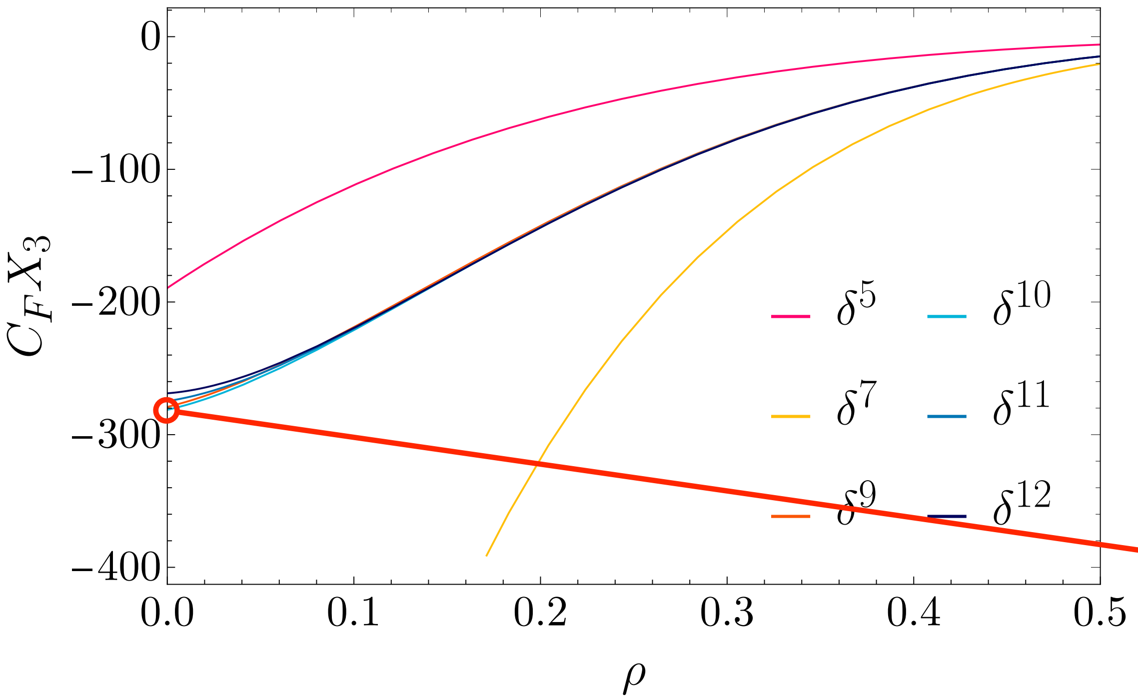
$$C_F X_3(\rho = 0.28) = -91.2 \pm 0.4\,(0.4\%)$$

MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

$$m_b^{\rm OS} : m_c^{\rm OS} \quad 1 - 1.78\left(\frac{\alpha_s}{\pi}\right) - 13.1\left(\frac{\alpha_s}{\pi}\right)^2 - 163.3\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm kin}(1\text{ GeV}) : \overline{m}_c(2\text{ GeV}) \quad 1 - 1.24\left(\frac{\alpha_s}{\pi}\right) - 3.65\left(\frac{\alpha_s}{\pi}\right)^2 - 1.0\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm 1S} : m_c \text{ via HQET} \quad 1 - 1.38\left(\frac{\alpha_s}{\pi}\right) - 6.32\left(\frac{\alpha_s}{\pi}\right)^2 - 33.1\left(\frac{\alpha_s}{\pi}\right)^3$$



MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 A_{\rm ew}}{192\pi^3} |V_{cb}|^2 \left(X_0(0) + C_F \sum_n \left(\frac{\alpha_s}{\pi}\right)^n X_n(0) \right)$$

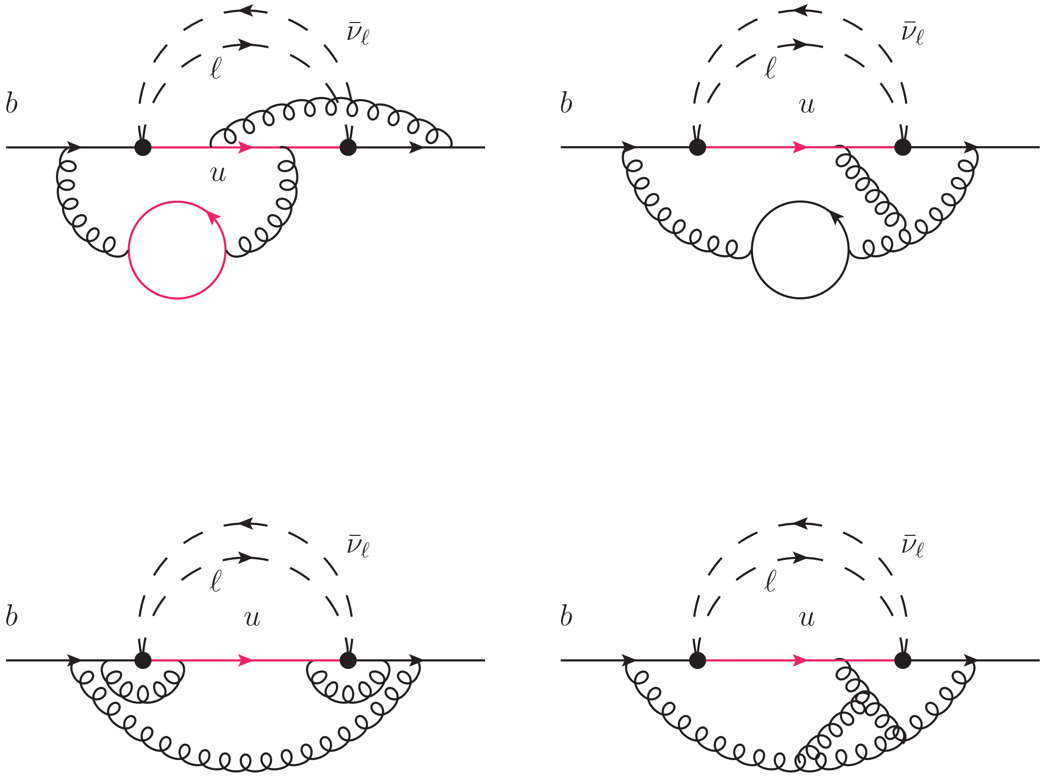
$$C_F X_3(\rho = 0) = -269 \pm 27\,(10\%)$$

$$m_b^{\rm kin}(1\ \text{GeV}) : \overline{m}_c(3\ \text{GeV}) \quad \Gamma_{\rm sl} \simeq 1 - 0.19\,|_{\alpha_s} + 0.019\,|_{\alpha_s^2} + 0.032\,(9)\,|_{\alpha_s^3}$$

- Underestimated uncertainty on X_3?
- Kinetic mass not good for $b \to u$?
- Weak-annihilation?
- Large m_c effects at $O(\alpha_s^3)$?

Third order correction to $b \to u$

MF, Usovitsch, hep-ph-2310.03685

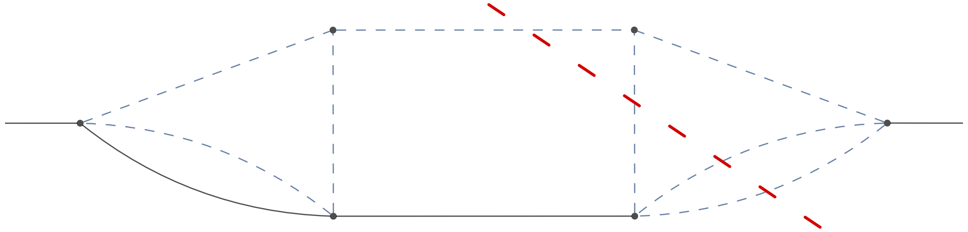


Fermionic corrections

$$
\begin{aligned}
X_3 = & \; N_L^2 T_F^2 X_{N_L^2} + N_H^2 T_F^2 X_{N_H^2} + N_H N_L T_F^2 X_{N_H N_L} \\
& + N_L T_F (C_F X_{N_L C_F} + C_A X_{N_L C_A}) \\
& + N_H T_F (C_F X_{N_H C_F} + C_A X_{N_H C_A}) \\
& + C_F^2 X_{C_F^2} + C_F C_A X_{C_F C_A} + C_A^2 X_{C_A^2}
\end{aligned}
$$

Bosonic corrections

Numerical evaluation of 5 loop integrals



- Challenging integration-by-parts reduction
- **1369** master integrals
- Numerical evaluation with 40 digits with AMFlow
  Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565
- Parallel calculation large-N_c contributions
  Chen, Li, Li, Wang, Wand, Wu, hep-ph/2309.00762

	This work	Ref. [28]	Difference
$T_F^2 N_L^2$	-6.9195	-6.34 (42)	8.3%
$T_F^2 N_H^2$	-1.8768×10^{-2}	$-1.97\,(42) \times 10^{-2}$	5.0%
$T_F^2 N_H N_L$	-1.2881×10^{-2}	$-1.1\,(1.1) \times 10^{-2}$	12%
$C_F T_F N_L$	-7.1876	$-5.65\,(55)$	22%
$C_A T_F N_L$	42.717	39.7 (2.1)	7%
$C_F T_F N_H$	2.1098	2.056 (64)	2.5%
$C_A T_F N_H$	-0.45059	$-0.449\,(18)$	0.4%

$$C_F X_3 = 280.2 \quad \text{fermionic}$$
$$-536.4 \quad \text{bosonic, large } N_c$$
$$-11.6\,(2.7) \quad \text{bosonic, subleading } N_c$$
$$= -267.8\,(2.7)$$
MF, Usovitsch, hep-ph-2310.03685

$$C_F X_3(\rho = 0) = -269 \pm 27\,(10\%)$$

MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

Conclusions

- In the last years the theory of inclusive decays has greatly profited from developments in computational methods for multi-loop integrals.

- Achievements in the last 2-3 years: kinetic mass, total rate, uncut moments at $O(\alpha_s^3)$

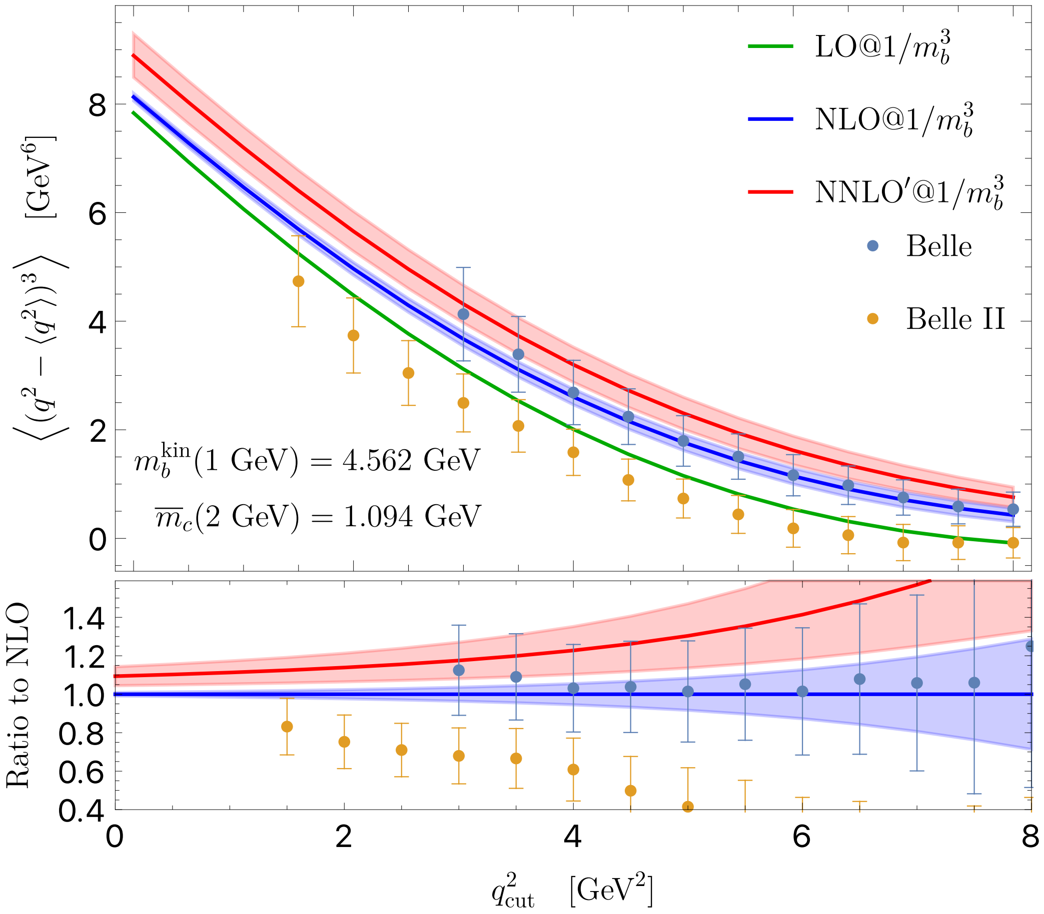
- Differential q^2 spectrum at NNLO analytic:
  - $\langle q^{2n} \rangle_{q^2_{\rm cut}}$
  - but also $\mathrm{Br}(q^2_{\rm cut})$, $\langle M_X^{2n} \rangle_{q^2_{\rm cut}}$ and $\langle E_l^n \rangle_{q^2_{\rm cut}}$

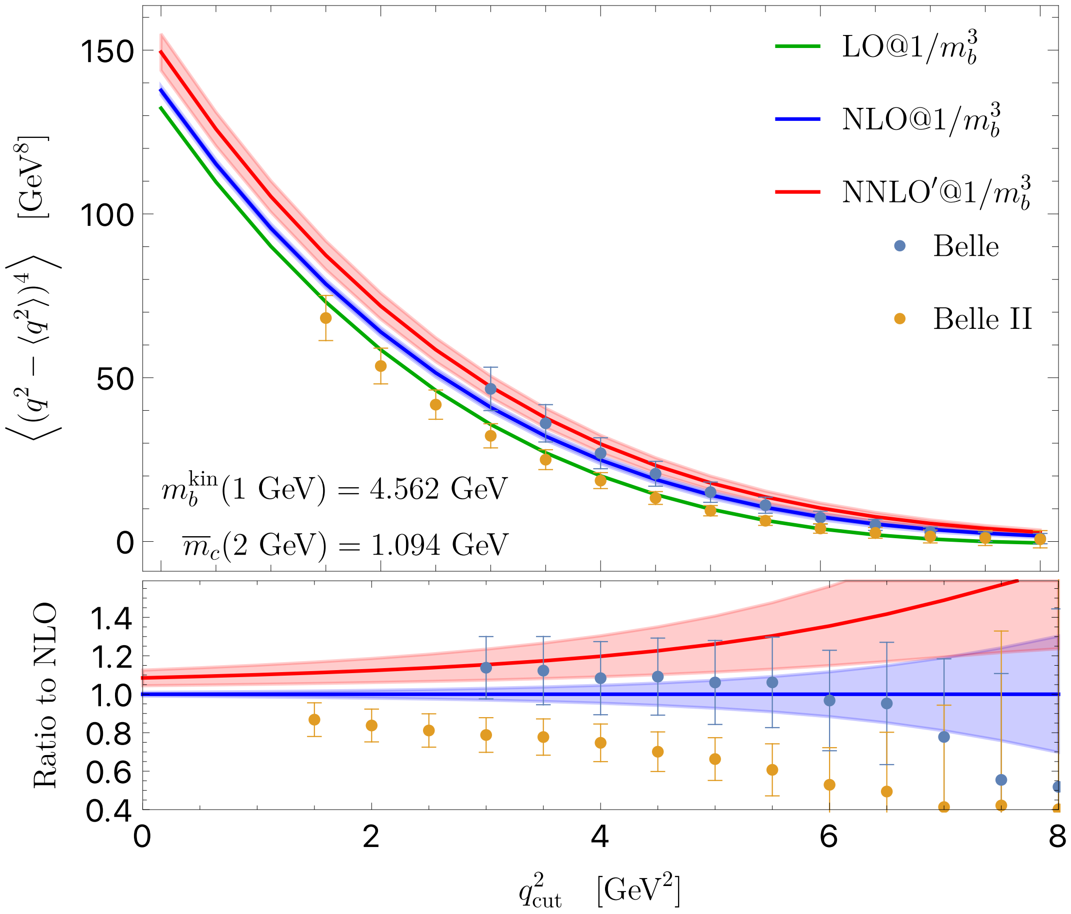
- Differential E_l spectrum likely also doable at NNLO analytically
- NLO correction to power suppressed terms (ρ_D and $1/m_b^{4,5}$) may be obtained with the same strategy

Conclusions

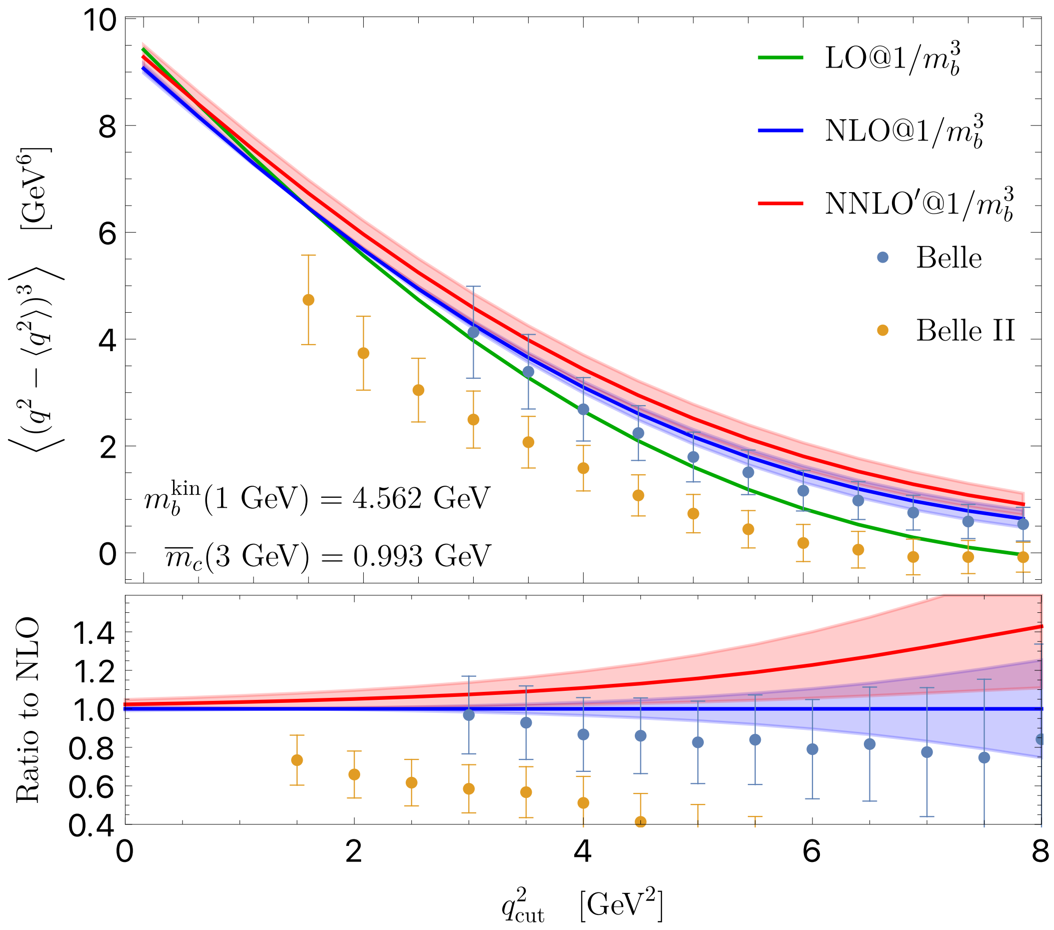
- Kinetic mass at $O(\alpha_s^4)$: reduce $\overline{\mathrm{MS}} \to \mathrm{kin}$ scheme conversion uncertainty on m_b from 15 MeV to ~ 8 MeV
- Power correction of order μ_{WC}^3/m_b^3 in the kinetic mass definition
- We are working hard to be ready for the new upcoming measurements at Belle II!

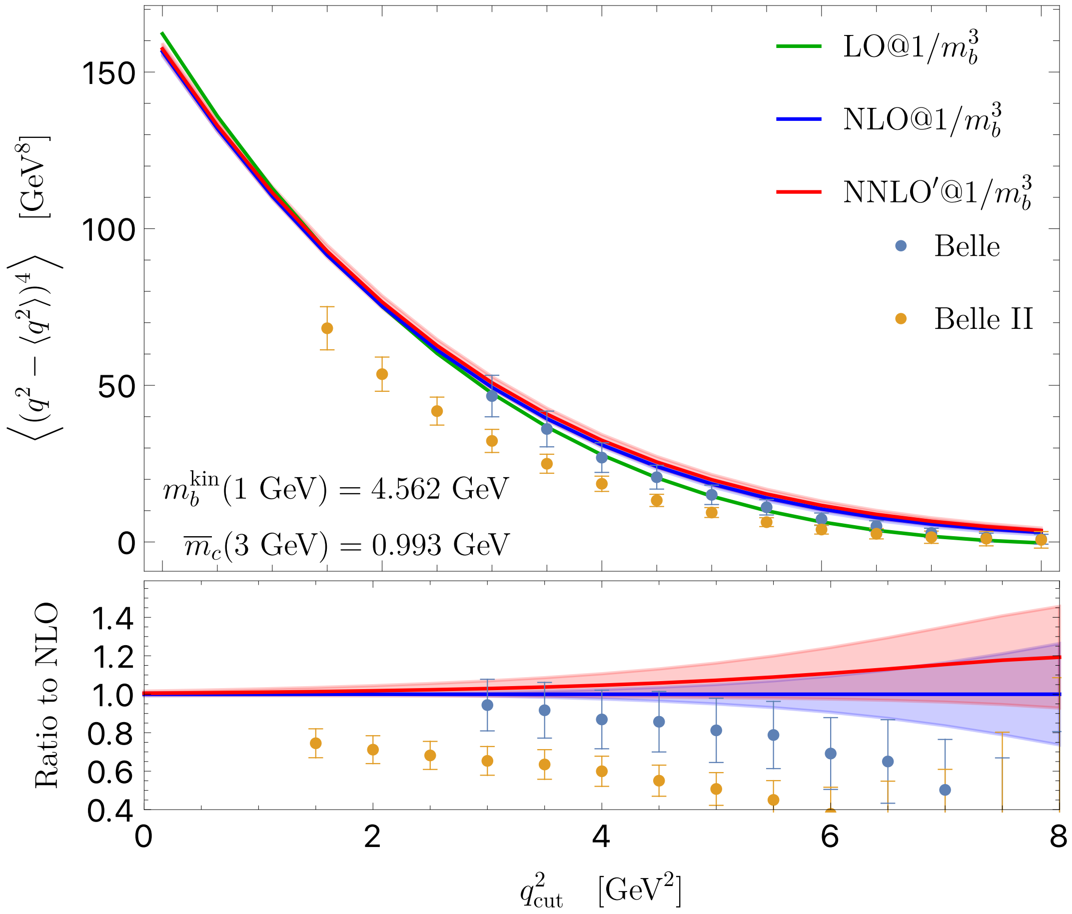
CERN





HQE parameters from: Bernlochner et al, JHEP 10 (2022) 068





HQE parameters from: Bernlochner et al, JHEP 10 (2022) 068

Kinetic scheme

- Calculation easily performed in the pole mass scheme for m_b and m_c
- Pole scheme leads to badly behaved perturbation theory
- The fits of $|V_{cb}|_{\text{inc}}$ use the kinetic scheme

$$m_b^{\text{pole}} = m_b^{\text{kin}}(\mu) + [\overline{\Lambda}(\mu)]_{\text{pert}} + \frac{[\mu_\pi^2(\mu)]_{\text{pert}}}{m_b^{\text{kin}}(\mu)} + \dots$$

- Also the HQE parameters are re-defined in the kinetic scheme

$$\mu_\pi^2(\mu) = \mu_\pi^2(0) - [\mu_\pi^2(\mu)]_{\text{pert}} \qquad \rho_D^3(\mu) = \rho_D^3(0) - [\rho_D^3(\mu)]_{\text{pert}}$$

$$[\mu_\pi^2(\mu)]_{\text{pert}} = \frac{\alpha_s}{\pi} C_F \mu^2 + \dots$$

$$[\rho_D^3(\mu)]_{\text{pert}} = \frac{2}{3}\frac{\alpha_s}{\pi} C_F \mu^3 + \dots$$

$$m_b^{\text{kin}}(1\,\text{GeV}) = 4.573\,\text{GeV} \quad \overline{m}_c(2\,\text{GeV}) = 1.092\,\text{GeV} \quad q^2_{\text{cut}} = 2\,\text{GeV}^2$$

$$\langle q^2 \rangle = 5.853470329 + 0.2365684899\,\text{api CF} - 5.870252829\,\text{api}^2\,\text{CF} + 12.31040061\,\text{api}^2\,\text{CF flagb0}$$

$$\langle (q^2 - \langle q^2 \rangle)^2 \rangle = 6.599610658 + 0.5418960093\,\text{api CF} - 20.80577919\,\text{api}^2\,\text{CF} + 48.85002521\,\text{api}^2\,\text{CF flagb0}$$

$$\langle (q^2 - \langle q^2 \rangle)^3 \rangle = 7.003374353 + 3.90484573\,\text{api CF} - 45.2523971\,\text{api}^2\,\text{CF} + 149.8472321\,\text{api}^2\,\text{CF flagb0} - 17$$

$$m_b^{\text{kin}}(1\,\text{GeV}) = 4.573\,\text{GeV} \quad \overline{m}_c(3\,\text{GeV}) = 0.993\,\text{GeV} \quad q^2_{\text{cut}} = 2\,\text{GeV}^2$$

$$\langle q^2 \rangle = 6.072411413 - 0.7043840353\,\text{api CF} - 13.53637271\,\text{api}^2\,\text{CF} + 10.36154604\,\text{api}^2\,\text{CF flagb0}$$

$$\langle (q^2 - \langle q^2 \rangle)^2 \rangle = 7.448558317 - 3.423938153\,\text{api CF} - 51.49139191\,\text{api}^2\,\text{CF} + 43.98509335\,\text{api}^2\,\text{CF flagb0} -$$

$$\langle (q^2 - \langle q^2 \rangle)^3 \rangle = 8.740268785 - 4.734517097\,\text{api CF} - 116.2284133\,\text{api}^2\,\text{CF} + 157.5509862\,\text{api}^2\,\text{CF flagb0} -$$