Semileptonic decay rate for $B \to D\,\ell\,\bar{\nu}_\ell$

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1 Overview

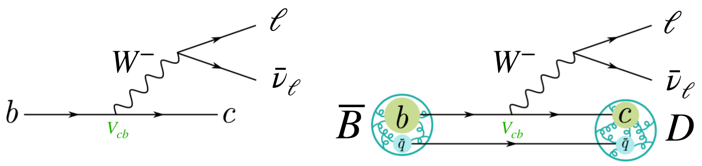

Fig. 1: Quark and parton-level decay of $B \to D\,\ell\,\bar{\nu}_\ell$ are shown.

The starting point for the calculation of the $1 \to 3$ decay rate is Fermi's golden rule:

$$\mathrm{d}\Gamma = \frac{1}{2m_B}\,|\mathcal{M}|^2\,\mathrm{d}\Pi_3\,, \tag{1}$$

with $\mathcal{M}$ denoting the matrix element we wish to calculate, $\mathrm{d}\Pi_3$ the Lorentz invariant phase space or short LIPS, and m_B further denoting the B-meson mass. The $b \to c\,\ell\bar{\nu}_\ell$ process at quark level is shown in Figure 1 and its matrix element is given by

$$\mathcal{M} = \left[\bar{u}\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma_5)v\right]\frac{ig_{\mu\nu} - q_\mu q_\nu/m_W^2}{q^2 - m_W^2}\left[\bar{c}\,V_{cb}\frac{-ig_W}{2\sqrt{2}}\gamma^\nu(1-\gamma_5)\,b\right]. \tag{2}$$

Here $\bar{u}$, v, c, and b denote the spinors of the charged lepton, the neutrino, the charm quark, and the bottom quark, respectively. The other terms are: $q^2 = q^\mu q_\mu$ denotes the four-momentum transfer to the lepton-neutrino pair in the decay, m_W denotes the W-bosons mass, V_{cb} denotes the $b \to c$ CKM matrix element, and γ^μ are the usual gamma matrices with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Note that $\frac{1}{2}(1-\gamma_5)$ acts as a left-handed projection operator to ensure the $V-A$ nature of the SM charged weak current.

At low energies ($q^2 \ll m_W^2$) the W-boson propagator, $\frac{ig_{\mu\nu}-q_\mu q_\nu/m_W^2}{q^2-m_W^2}$, can be simplified by neglecting the q^2 and $q_\mu q_\nu$ terms, as they are orders of magnitudes smaller than the W-boson mass squared. In semileptonic B-meson decays involving charmed final states the allowed values for q^2 range from $m_\ell^2 \approx 0\ \mathrm{GeV}^2$ to about $11.6\ \mathrm{GeV}^2$, whereas $m_W^2 \approx 6400\ \mathrm{GeV}^2$. Introducing Fermi's constant to describe the weak interaction as an effective coupling, $G_F = \frac{\sqrt{2}g_W^2}{8m_W^2}$, simplifies the matrix element to

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}V_{cb}\,\underbrace{[\bar{u}\gamma^\mu(1-\gamma_5)v]}_{L^\mu}\,\underbrace{[\bar{c}\gamma^\nu(1-\gamma_5)\,b]}_{h^\mu}\,. \tag{3}$$

The b- and c-quarks interact non-perturbatively with the spectator quark of the B- and final state mesons. The simplest decay is to study the decay of a B-meson into a pseudoscalar final state, e.g. a D (the eventual expression will be identical also for π with the replacement $V_{cb} \to V_{ub}$) and in what follows we will study the decay of $B \to D\,\ell\,\bar{\nu}_\ell$ with ℓ being a light lepton whose mass we can neglect ($m_\ell \approx 0$). This is an excellent approximation for electrons and muons. To make this apparent we introduce the wave function of the B- and D-meson as

$$\langle D(p_D)|h_\mu|B(p_B)\rangle = H_\mu \,, \tag{4}$$

with p_B denoting the four-momentum of the B-meson and p_D the four-momentum of the D-meson. The matrix element then becomes the product of a hadronic current H^μ and leptonic current L_μ:

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}} V_{cb}\, L_\mu H^\mu \,. \tag{5}$$

2 Reference frames

Before discussing further L_μ and H_μ we briefly should discuss the kinematic properties of the $1 \to 3$ decay of $B \to D\,\ell\,\bar{\nu}_\ell$. Two frames are of particular importance: the frame in which the B-meson is at rest (experimentally accessible in analyses using hadronic tagging) and the restframe of the virtual W-boson. Before we start, let us define once more the 4-momentum of the W-boson and write out some relations with the B- and D-meson and the lepton-neutrino-pair:

$$q^\mu = p_W^\mu = p_B^\mu - p_D^\mu = p_\ell^\mu + p_\nu^\mu \,, \qquad \text{and} \qquad q^2 = q_\mu q^\mu \,:\, \text{mass squared of the virtual } W \text{ boson.} \tag{6}$$

2.1 W-frame

Without loss of any generality one can choose the direction of each momentum such that the final calculations will be most convenient. This is guaranteed as one can always rotate in set of coordinates that fulfill this specific choice. Our choice is shown in Figure 2: the B-meson propagates along the negative z-axis, and we choose the $y-x$-plane as the decay plane, i.e. the lepton and neutrino propagate back-to-back in $y-x$ direction. As the virtual W-boson is at rest,

$$\vec{p}_\ell = -\,\vec{p}_\nu \,, \tag{7}$$

and assuming zero lepton mass we find

$$p_\ell^\mu = \begin{pmatrix} |\vec{p}_\ell| \\ \vec{p}_\ell \end{pmatrix} , \qquad p_\nu^\mu = \begin{pmatrix} |\vec{p}_\nu| \\ \vec{p}_\nu \end{pmatrix} = \begin{pmatrix} |\vec{p}_\ell| \\ -\vec{p}_\ell \end{pmatrix} . \tag{8}$$

We further find that

$$q^2 = (p_\ell + p_\nu)^\mu (p_\ell + p_\nu)_\mu = 4\,|\vec{p}_\ell|^2 \quad \to \quad |\vec{p}_\ell| = \frac{q}{2} \,. \tag{9}$$

The 3-momentum of the B-meson is completely passed to the D-meson,

$$\vec{p}_B = \vec{p}_D \,, \tag{10}$$

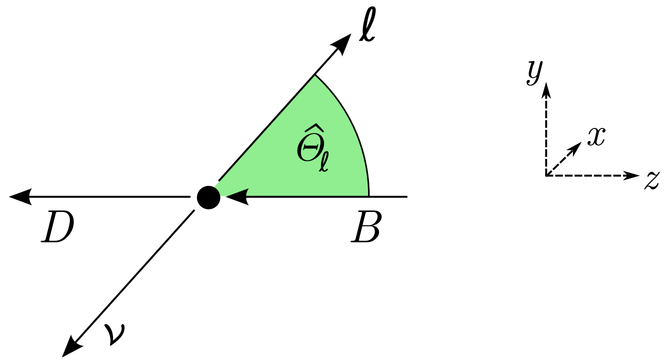

Fig. 2: The kinematics in the W-frame are shown: the charged lepton and neutrino are back-to-back, which is only possible if the D-meson and the B-meson have identical 3-momenta.

and thus

$$p_B^\mu = \begin{pmatrix} E_\ell + E_\nu + E_D \\ \vec{p}_D \end{pmatrix} = \begin{pmatrix} q + E_D \\ \vec{p}_D \end{pmatrix}, \tag{11}$$

where we used that $q = E_\ell + E_\nu$ and the W-boson is at rest. The invariant mass of the B-Meson can be used to find a useful relation between q^2 and $|\vec{p}_D|$:

$$m_B^2 = p_B^2 = q^2 + 2qE_D + E_D^2 - |\vec{p}_D|^2 = q^2 + 2qE_D + m_D^2 \,. \tag{12}$$

Solving for E_D results in $E_D = \frac{m_B^2 - m_D^2 - q^2}{2q}$ and we find

$$|\vec{p}_D| = \sqrt{E_D^2 - m_D^2} = \sqrt{\frac{\left(m_B^2 - m_D^2 - q^2\right)^2}{4q^2} - m_D^2} \,. \tag{13}$$

2.2 B-frame

This will be discussed as classwork next week Wednesday. There you will derive the relation of

$$|\vec{p}_D| = \frac{m_B}{q} |\tilde{\vec{p}}_D| \,, \tag{14}$$

with $\tilde{\vec{p}}_D$ denoting the D-meson 3-momentum in the B-meson restframe.

3 The squared matrix element

To evaluate the rate, we need to calculate

$$|\mathcal{M}|^2 = \frac{G_F^2 \left|V_{cb}\right|^2}{2} L^\mu L^{\rho *} H_\mu H_\rho^* \,, \tag{15}$$

and we will discuss the leptonic matrix $L^\mu L^{\rho *}$ and the hadronic matrix $H_\mu H_\rho^*$ separately. It is convenient to evaluate both in the W-restframe as many of the expressions vanish.

3.1 The leptonic matrix: $L^\mu L^{\rho *}$

The leptonic matrix is given by

$$L^\mu L^{\rho *} = \left[\bar{u}\gamma^\mu(1-\gamma_5)v\right]\left[\bar{u}\gamma^\rho(1-\gamma_5)v\right]^* \,, \tag{16}$$

and contains multiple spin configurations. As we are interested in the spin-summed expressions we can use the trace formalism to express the leptonic matrix as a single trace of γ matrices:

$$L^\mu L^{\rho *} = \mathrm{Tr}\left[\gamma^\mu(1-\gamma_5)\left(\not{p}_\nu - m_\nu\right)\gamma^0\left(\gamma^\rho(1-\gamma_5)\right)^\dagger\gamma^0\left(\not{p}_\ell + m_\ell\right)\right] \,. \tag{17}$$

This trace can be simplified to

$$L^\mu L^{\rho *} = \left(p_\nu\right)_\alpha \left(p_\ell\right)_\beta \mathrm{Tr}\left[\gamma^\mu(1-\gamma_5)\gamma^\alpha\gamma^0(1-\gamma_5)\gamma^{\rho\dagger}\gamma^0\gamma^\beta\right] \,, \tag{18}$$

by assuming $m_\ell = m_\nu = 0$. This trace can be further reduced by using $\mathrm{Tr}(A+B) = \mathrm{Tr}(A)+\mathrm{Tr}(B)$ and the anti-commutation property of $\{\gamma^\mu, \gamma_5\} = 0$ to

$$L^\mu L^{\rho *} = \left(p_\nu\right)_\alpha \left(p_\ell\right)_\beta \left\{2\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\rho\gamma^\beta\right] + 2\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\alpha\gamma^\rho\gamma^\beta\right]\right\} \tag{19}$$

and the remaining traces can be simplified to

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\rho\gamma^\beta\right] = 4\left(g^{\mu\alpha}g^{\rho\beta} - g^{\mu\rho}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\rho}\right) \,, \tag{20}$$

$$\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\alpha\gamma^\rho\gamma^\beta\right] = -4i\,\varepsilon^{\mu\alpha\rho\beta} \,, \tag{21}$$

with $\varepsilon^{\mu\alpha\rho\beta}$ denoting the Levi-Civita tensor. We thus find

$$L^\mu L^{\rho *} = 8\left(p_\nu\right)_\alpha \left(p_\ell\right)_\beta \left(g^{\mu\alpha}g^{\rho\beta} - g^{\mu\rho}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\rho} - i\,\varepsilon^{\mu\alpha\rho\beta}\right) \,, \tag{22}$$

$$= 8\left[\left(p_\nu\right)^\mu \left(p_\ell\right)^\rho - g^{\mu\rho}p_\nu p_\ell + \left(p_\ell\right)^\mu \left(p_\nu\right)^\rho\right] - 8i\left(p_\nu\right)_\alpha \left(p_\ell\right)_\beta \varepsilon^{\mu\alpha\rho\beta} \,. \tag{23}$$

We will now investigate the leptonic matrix in the W-boson restframe and study its temporal, mixed temporal-spatial, and its spatial components. In the restframe of the virtual W-boson we showed that

$$\left(p_\ell\right)^0 = \left(p_\nu\right)^0 = \frac{q}{2} \,, \qquad \vec{p}_\ell = -\vec{p}_\nu \,, \qquad \vec{p}_\ell \cdot \vec{p}_\nu = -\frac{q^2}{4} \,, \qquad p_\nu p_\ell = \frac{q^2}{2} \,. \tag{24}$$

Temporal component: $L^0 L^{0*}$**:** Inserting the expressions from above we find

$$L^0 L^{0*} = 8\left[\left(p_\nu\right)^0 \left(p_\ell\right)^0 - g^{00}p_\nu p_\ell + \left(p_\ell\right)^0 \left(p_\nu\right)^0\right] - 8i\left(p_\nu\right)_\alpha \left(p_\ell\right)_\beta \varepsilon^{0\alpha 0\beta} \,. \tag{25}$$

$$= 8\left(\frac{q^2}{4} - \frac{q^2}{2} + \frac{q^2}{4}\right) - 0 = 0 \,, \tag{26}$$

i.e. the component vanishes in the W-restframe.

Mixed components: $L^0 L^{i*}$**:** Here $i = 1,2,3$ and inserting the expressions from above we find

$$L^0 L^{i*} = 8\left[\left(p_\nu\right)^0 \left(p_\ell\right)^i - \underbrace{g^{0i}p_\nu p_\ell}_{=0} + \left(p_\ell\right)^0 \left(p_\nu\right)^i\right] - 8i\left(p_\nu\right)_\alpha \left(p_\ell\right)_\beta \varepsilon^{0\alpha i\beta} \,, \tag{27}$$

$$= \underbrace{8\left[\left(p_\ell\right)^0 \left(p_\ell\right)^i - \left(p_\ell\right)^0 \left(p_\ell\right)^i\right]}_{=0} - 8i\left(p_\nu\right)_\alpha \left(p_\ell\right)_\beta \varepsilon^{0\alpha i\beta} \,, \tag{28}$$

$$= -8i\left(p_\nu\right)_\alpha \left(p_\ell\right)_\beta \varepsilon^{0\alpha i\beta} \,. \tag{29}$$

Inspecting the Levi-Civita tensor and the 3-momenta in the W-restframe further, we notice that

$$(p_\ell)_1 = (p_\nu)_1 = 0\,, \qquad (p_\ell)_2 = -\,(p_\nu)_2\;, \qquad (p_\ell)_3 = -\,(p_\nu)_3\;, \tag{30}$$

i.e. for $\alpha = 1$ or $\beta = 1$ the expression vanishes and thus $L^0 L^{2*} = L^0 L^{3*} = 0$. Finally the last term $L^0 L^{1*}$ we find

$$L^0 L^{1*} = -8i\left[(p_\nu)_3\,(p_\ell)_2\,\underbrace{\varepsilon^{0312}}_{=+1} + (p_\nu)_2\,(p_\ell)_3\,\underbrace{\varepsilon^{0213}}_{=-1}\right], \tag{31}$$

$$= -8i\left[-\,(p_\ell)_3\,(p_\ell)_2 + (p_\ell)_2\,(p_\ell)_3\right] = 0\,, \tag{32}$$

i.e. all mixed components of the lepton matrix vanish.

Spatial components: $L^i L^{j*}$**:** The spatial components with $i = 1,2,3$ and $j = 1,2,3$ are

$$L^i L^{j*} = 8\left[(p_\nu)^i\,(p_\ell)^j - g^{ij} p_\nu p_\ell + (p_\ell)^i\,(p_\nu)^j\right] - 8i\,(p_\nu)_\alpha\,(p_\ell)_\beta\;\varepsilon^{i\alpha j\beta}\,, \tag{33}$$

$$= 8\left(-\frac{q^2}{4}e^i e^j + \delta^{ij}\frac{q^2}{2} - \frac{q^2}{4}e^i e^j\right) + 4\,i\,q^2\,e_k\varepsilon^{ijk}\,, \tag{34}$$

$$= 4q^2\left(\delta^{ij} - e^i e^j\right) + 4\,i\,q^2 e_k\varepsilon^{ijk} = 4q^2\left(\delta^{ij} - e^i e^j + i e_k\varepsilon^{ijk}\right), \tag{35}$$

where we introduced the unit vector in direction of the lepton, e^i, and reduced the 4-dimensional Levi-Civita tensor to a 3-dimensional expression via

$$(p_\nu)_\alpha\,(p_\ell)_\beta\;\varepsilon^{i\alpha j\beta} = (p_\nu)_0\,(p_\ell)_k\;\varepsilon^{i0jk} + (p_\nu)_k\,(p_\ell)_0\;\varepsilon^{ikj0}\,, \tag{36}$$

$$= -\,(p_\nu)_0\,(p_\ell)_k\;\varepsilon^{0ijk} + (p_\nu)_k\,(p_\ell)_0\;\varepsilon^{0ijk}\,, \tag{37}$$

$$= -(\frac{q}{2})\,(\frac{q}{2}e_k)\,\varepsilon^{0ijk} - (\frac{q}{2}e_k)\,(\frac{q}{2})\,\varepsilon^{0ijk} \tag{38}$$

$$= -2\frac{q^2}{4}\,e_k\,\varepsilon^{0ijk} = -\frac{q^2}{2}\,e_k\,\varepsilon^{ijk}\,. \tag{39}$$

3.2 The hadronic matrix: $H_\mu H_\rho{}^*$

The hadronic matrix contains non-perturbative physics, i.e. can in general not be calculated in a perturbative manner by starting from the quark-level diagram and include corrections of additional diagrams in a series of increasing α_s as here $\alpha_s > 1$. There are however, a range of properties that the hadronic current H_μ,

$$H_\mu = \langle D(p_D)|h_\mu|B(p_B)\rangle\,, \tag{40}$$

has to fulfill that help us understanding its general structure:

1. The unknown expression of the hadronic current needs to be Lorentz covariant, i.e. its transformation properties are identical with other entities we encounter in relativistic quantum field theories.

This might not sound like much, but due to this the expression of H_μ has to be proportional to the Lorentz vectors and scalars that enter the hadronic matrix element. Without the loss of generality we can thus expand the unknown hadronic current as

$$H_\mu = (p_B + p_D)_\mu\;f_+(p_B^2, p_D^2, p_B p_D) + (p_B - p_D)_\mu\;f_-(p_B^2, p_D^2, p_B p_D)\,. \tag{41}$$

The Lorentz vectors of the hadronic transition, $(p_B \pm p_D)_\mu$, are proportional to some unknown scalar functions $f_\pm$, which can only depend on Lorentz scalars of the matrix element. The only such scalars are p_B^2, p_D^2, $p_B p_D$. Note that another choice would have been

$$H_\mu = (p_B)_\mu \; f(p_B^2, p_D^2, p_B p_D) + (p_D)_\mu \; g(p_B^2, p_D^2, p_B p_D)\,, \tag{42}$$

but as we will see shortly the first Ansatz has some interesting features. We know more:

2. The matrix element describes an on-shell transition into an an on-shell transition; thus $p_B^2 = m_B^2$ and $p_D^2 = m_D^2$.

Thus $f_\pm(p_B^2 = m_B^2, p_D^2 = m_D^2, p_B p_D) = f_\pm(p_B p_D)$ does only depend on a single Lorentz scalar. The choice of $p_B p_D$ as the Lorentz scalar is not very convenient. A better (and now to you familiar) choice is

$$q^2 = (p_B - p_D)^2 = m_B^2 + m_D^2 - 2 p_B p_D\,, \tag{43}$$

and we thus write

$$H_\mu = (p_B + p_D)_\mu \; f_+(q^2) + (p_B - p_D)_\mu \; f_-(q^2)\,. \tag{44}$$

Note that we are allowed to do this, as we can always transform a function of $p_B p_D$ into a function of q^2. Before discussing more the a-priori unknown functions $f_\pm$, let us briefly investigate the Lorentz structure more in the W-boson restframe. We recall from the previous section that

$$L^\mu \, q_\mu = 0\,, \tag{45}$$

as in the W-restframe q has only temporal components whereas L^μ is only non-zero for spatial components. Thus the second term proportional to $(p_B - p_D)_\mu = q_\mu$ will vanish when contracted eventually with L^μ. Note that albeit we used the W-restframe which used the approximations that $m_\ell = 0$, this term does not vanish in general for $m_\ell \neq 0$. This can be seen independent of L^μ, by noticing that $(p_B - p_D)_\mu = (p_\ell + p_\nu)_\mu$. Using the equation of motion of the lepton and the neutrino one can show that these terms in the eventual contraction are proportional to $\not{p}_\ell + \not{p}_\nu = m_\ell - m_\nu$, i.e. vanish in the zero lepton mass limit. We will encounter f_- again in the next exercise sheet where we will study

$$R(D) = \frac{\Gamma(B \to D\,\tau\,\bar{\nu}_\tau)}{\Gamma(B \to D\,\ell\,\bar{\nu}_\ell)}\,. \tag{46}$$

Incorporating the above observation results in

$$H_\mu = (p_B + p_D)_\mu \; f_+(q^2)\,, \tag{47}$$

and we are left with a single unknown function f_+, which depends on q^2, that fully describes the matrix element in the $B \to D\,\ell\,\bar{\nu}_\ell$ decay for light leptons. This function is called the $B \to D$ form factor.

The $f_+(q^2)$ form factor: How can we calculate or predict $f_+(q^2)$? There are several methods:

1. Examine the QCD Lagrangian in the $m_{c,b} \to \infty$ limit: this results in an effective field theory called HQET (short for Heavy Quark Effective Field Theory) which systematically allows to write down $\mathcal{M}$ in orders of $m_{c,b}$ and α_s. As four-momenta are ill-defined entities in

the infinite mass limit, the form factor itself is parametrized via $w = v_B v_D = \frac{m_B^2 + m_D^2 - q^2}{2 m_B m_D}$, the product of the four-velocities of the two heavy quarks. The form factor f_+ is identical with the so-called Isgur-Wise function ζ and one can show

$$f_+(q^2_{\max}) = \zeta(w = 1) = 1\,. \tag{48}$$

(without proof) and as both quarks are heavy with respect to the spectator quark, we expect that it will not notice the $b \to c$ transition if the four-velocities do not change much.

$\to$ From this we expect the form factor to be a slowly varying function of w, and one could parametrize it as

$$f_+(q^2) = 1 + (w - 1)\rho_D\,, \tag{49}$$

with ρ_D encapsulating the non-perturbative dynamic.

2. Lattice QCD: Lattice QCD can calculate the form factor at large q^2.
3. QCD Sum-rules allow to calculate the form factor at low q^2.
4. Extract the form factor directly from data. I.e. measure the decay rate as a function of q^2 and fit with a model independent parametrization.

Today a combination of all these approaches is common. To extract $|V_{cb}|$ from a measured branching fraction approach 4. will need to receive input from either 1-3 as $|V_{cb}|$ is degenerate with the form factor normalization.

The hadronic current in the W-frame: We can further simplify the expression derived in the W restframe. We already saw that contractions of the form

$$L^\mu H_\mu\,, \tag{50}$$

will be only non-zero if μ is spatial. The spatial parts of the hadronic current are

$$\begin{aligned} \vec{H} &= (\vec{p}_B + \vec{p}_D)\, f_+(q^2) = 2\vec{p}_D f_+(q^2)\,, && (51)\\ &= -2|\vec{p}_D|\vec{e}_z f_+(q^2)\,, && (52)\\ &= -2\frac{m_B}{q}|\tilde{\vec{p}}_D|\vec{e}_z f_+(q^2)\,, && (53) \end{aligned}$$

where we introduced $\vec{e}_z$ the unit vector of the $z - axis$ and $|\tilde{\vec{p}}_D|$ is the absolute value of the D-meson momentum in the B-frame (cf. Section 2.2).

3.3 Putting everything together: the complete Matrix element

We now derived all relevant expressions and can put the complete matrix element together. We find

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{G_F^2\, |V_{cb}|^2}{2} 4\, q^2 \left(\delta^{ij} - e^i e^j + i e^k \varepsilon^{ijk}\right) 4\, f_+^2(q^2) \frac{m_B^2}{q^2} |\tilde{\vec{p}}_D|^2\, (e_z)_i\, (e_z)_j\,, && (54)\\ &= 8\, G_F^2\, |V_{cb}|^2\, m_B^2\, |\tilde{\vec{p}}_D|^2 f_+^2(q^2) \left(1 - \cos\theta_\ell^2\right)\,, && (55) \end{aligned}$$

where we used that $e^i\,(e_z)_i = \cos\theta_\ell$ (cf. Figure 2) and $e_z^1 = e_z^2 = 0$ what ensures $ie^k\varepsilon^{ijk}$ to vanish. We now only need to multiply in the LIPS for a three-body decay

$$\mathrm{d}\Pi_3 = \left(\frac{1}{(4\pi)^5}\frac{|\tilde{\vec{p}}_D|}{m_B}\right)\mathrm{dq}^2\mathrm{d}\Omega_\mathrm{D}\,\mathrm{d}\Omega_\ell\,, \tag{56}$$

as we have no angular dependence on the D-meson and the dependence in the matrix element for the lepton is $1-\cos^2\theta_\ell = \sin^2\theta_\ell$ we end up with (without proof)

$$\int\!\!\int \sin^2\theta_\ell \mathrm{d}\Pi_3\mathrm{d}\,\Omega_D\,\mathrm{d}\Omega_\ell = \frac{8\pi}{3}\left(\frac{1}{(4\pi)^4}\frac{|\tilde{\vec{p}}_D|}{m_B}\right)\mathrm{dq}^2\,. \tag{57}$$

The total $B\to D\,\ell\,\bar{\nu}_\ell$ decay rate for $m_\ell = 0$ as a function of q^2 is thus given

$$\boxed{\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}\right|_{m_\ell=0} = \frac{G_F^2\,|V_{cb}|^2}{24\pi^3}\,|\tilde{\vec{p}}_D|^3\left[f_+^2(q^2)\right]\,.} \tag{58}$$

Incorporating the full lepton mass effects this expression becomes (without proof),

$$\boxed{\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}\right|_{m_\ell\neq0} = \frac{G_F^2\,|V_{cb}|^2}{24\pi^3}\,|\tilde{\vec{p}}_D|^3\left(1-\frac{m_\ell^2}{q^2}\right)^2\left[f_+^2(q^2)\left(1+\frac{m_\ell^2}{2q^2}\right)+\frac{3}{2}\frac{m_\ell^2}{q^2}\left(\frac{m_B^2-m_D^2}{2m_B|\tilde{\vec{p}}_D|}\right)f_0^2(q^2)\right]\,.} \tag{59}$$

with $f_0(q^2)$ denoting a form factor related to $f_-(q^2)$.

3.4 The ratio $R(D)$:

This allows us to write down an expression for $R(D)$ in the SM

$$R(D) = \frac{\int \mathrm{d}q^2\left[\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}\right|_{m_\ell=m_\tau}\right]}{\int \mathrm{d}q^2\left[\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}\right|_{m_\ell=0}\right]}\,. \tag{60}$$

Note that the dependence on $|V_{cb}|^2$ and G_F^2 cancel in the ratio and that the numerator depends on both form factors $f_{+/0}$ and the denominator only on f_+. This will lead to a certain cancellation of uncertainties. To make a prediction one has to determine the form factor f_+, either by fitting the $B\to D\,e\,\bar{\nu}_e$ and $B\to D\,\mu\,\bar{\nu}_\mu$ q^2 spectra or by combining lattice and Sum rule constraints. The form factor f_0 can be constrained either through QCD relations (HQET relates f_+ and f_0) or by using results from the Lattice, as it cannot be directly measured using light leptons. We will do a homework exercise about this.